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MEETING OF 1940 NOVEMBER 8

Professor H. C. Plummer, President, in the Chair

The election by the Council of the following Fellow was duly confirmed:—

Rev. William Kilshaw, St. John's Vicarage, Guernsey, C.I.

The following candidates were proposed for election as Fellows of the Society, the name of the proposers from personal knowledge being appended:—

Norman Douglas Anderson, Master Mariner, Scindia Steam Navigation Co., Bombay, India (proposed by W. H. Coombs);

Brian Percival Mullins, B.Sc., 230 Broadwalk, Blackheath, London, S.E. 3 (proposed by S. B. Mattey);

Clement Frederick Press, Wharf Superintendent, Federated Malay States Railways, Port Swettenham, Selangor, F.M.S. (proposed by W. H. Coombs);

Hyman S. Spigl, Acting Government Astronomer of Western Australia, Perth, W.A. (proposed by A. D. Ross); and

Gordon W. Wares, Brenau College, Gainsville, Georgia, U.S.A. (proposed by S. Chandrasekhar).

One hundred and thirty-eight presents were announced as having been received since the last meeting, including:

H.M. Nautical Almanac Office, Apparent Places of Fundamental Stars, 1941;

H. Macpherson, A Biographical Dictionary of Astronomers (presented by the author).

The President made the following announcement:—

"For the fourth time during my term of office I have to announce the death of a Past President of the Society—Professor Alfred Fowler died on June 24 last. The following resolution was passed by the Council at their meeting this morning:—

The Council of the Royal Astronomical Society desires to place on record its sense of the great loss sustained by the Society through the death of Alfred Fowler,

Fellow of the Society since 1889, who filled the offices of Secretary, Foreign Secretary and President. The Council recognizes with deep appreciation the debt which astronomy owes to Professor Fowler for his outstanding work on solar and astrophysical research."

The President further announced that the Council had decided to cancel the Ordinary Meetings of the Society in December and January, owing to the difficulties of transport during the winter months, and to postpone the Annual General Meeting until 1941 March; no decision had been taken as to whether a meeting should be held in February or not. He informed Fellows that Council had decided, in principle, to hold Ordinary Meetings during the summer months rather than in the winter months, but suggested that any Fellow with definite views about a February meeting should inform the Assistant Secretary accordingly.

THE OBSERVATIONAL DETERMINATION OF THE LINE ABSORPTION COEFFICIENT AND ITS VARIATION WITH DEPTH IN THE SOLAR ATMOSPHERE

H. H. Plaskett

Introduction.—Investigations of the formation of absorption lines have primarily been concerned with predicting the residual intensity in a line for some hypothetical variation of the atomic absorption coefficient with depth in the solar or stellar atmosphere. In his basic paper Eddington * considered three main cases: first, that of the line absorption coefficient independent of depth which led to Milne's † formulation of the equation of transfer and its immediate integration; secondly, that of a two- or three-layer atmosphere, within each layer of which the line absorption coefficient was constant; and finally, that of a continuous variation of his quantity η, the ratio of line to continuous absorption coefficient, in accordance with the relation $1/(1+\eta)=\alpha+\beta\tau$, which led to the integration of the equation of transfer in terms of Bessel functions. Following upon an observational investigation ‡ which showed that the line absorption coefficient was not independent of optical depth, and a stimulating general discussion of the problem by ten Bruggencate §, substantial progress has been made in predicting line intensities for a number of modes of variation of the absorption coefficient with depth. Noteworthy amongst these are the cases discussed by Unsöld ||, Strömgren ¶ and Spitzer.**

The development of absorption-line theory has thus closely followed the method used by Schwarzschild †† in his classical investigation of the darkening to the limb in the continuous spectrum. He constructed an idealized *mathematical model*, with a fixed hypothetical element, namely the assumed validity of the laws of radiation leading to the formulation of the equation of radiative transfer, and with a variable hypothetical element, namely the mode of heat transfer. Depending upon the mode of heat transfer assumed, so the model predicts a specific darkening towards the limb; the predicted darkening which most closely mimics that observed then allows us to infer which is the dominant mode of transfer in the solar atmosphere.

While the mathematical model is probably the most powerful tool of the astrophysicist, there are cases where it may usefully be supplemented by the *observational model.* Both models are alike in any particular case in having the same fixed hypothetical element, but the observational differs from the mathematical model in replacing the variable hypothetical element by the observed phenomenon itself. Thus by means of an integral equation the observed phenomenon may be used to calculate the unknown functional variation of the parameter in which we are interested. In the case of the continuous spectrum ‡‡ this parameter is the specific intensity of the thermal radiation as a function of optical depth; in the case of the line spectrum the parameter is the absorption coefficient, also as a function of optical depth. The advantages of the observational model are that it yields, within the observational error, the actual march of the interesting variable, not some approximation to it chosen because of mathematical

* A. S. Eddington, *M.N.*, **89**, 620, 1929.
† E. A. Milne, *Phil. Trans.*, A, **223**, 201, 1922.
‡ H. H. Plaskett, *M.N.*, **91**, 870, 1931.
§ P. ten Bruggencate, *Zs. f. Ap.*, **4**, 159, 1932.
|| A. Unsöld, *Zs. f. Ap.*, **4**, 339, 1932. *Cf.* M. Minnaert, *Zs. f. Ap.*, **12**, 313, 1936.
¶ B. Strömgren, *Ap. J.*, **86**, 1, 1937. ** Lyman Spitzer, jun., *Ap. J.*, **87**, 1, 1939.
†† K. Schwarzschild, *Göttinger Nachrichten*, p. 41, 1906.
‡‡ H. H. Plaskett, *M.N.*, **96**, 418, 1936; P. Parchomenko, *Observatory*, **69**, 375, 1936.

simplicity, and, secondly, that in those cases, such as the heat transfer in the solar atmosphere, where the observed phenomenon is produced by more than one mechanism, the observational model indicates not merely the dominant mechanism, but also the contribution of each mechanism to the observed result.

The essential problems to be solved in constructing an observational model for absorption lines have been clearly indicated by ten Bruggencate *, and two of the three necessary integral equations actually given. It is the purpose of the present paper first to set out with no simplifying approximations the theory for this observational model in a form suitable for numerical calculation, and, secondly, to apply this analysis to the observed variation of the *Mg b* lines across the solar disc.

Section 1.—The Observational Model for Absorption Lines

1·1. *The Equation of Transfer.*—Let the mass coefficient of line absorption be $k_\lambda + s_\lambda$, where k_λ is the part producing true absorption, and s_λ the part producing scattering. Also let k_0 be the mass coefficient of continuous absorption, appropriate to the same wave-length λ. Similarly let j_λ be the mass coefficient of thermal line emission, and j_0 the mass coefficient of continuous emission at the same wave-length. These various mass coefficients are all functions of depth. If the specific intensity of the radiation field at the depth x is $I_\lambda(x, \theta)$, where θ is the angle between the direction of the flux considered and the outgoing normal, then considering the gains and losses experienced by the flux, confined within a small solid angle $d\omega$, which passes normally through a small cylinder of unit cross-section and thickness $dx \sec\theta$, we obtain the macroscopic equation of transfer

$$\cos\theta \frac{dI_\lambda(x, \theta)}{\rho dx} = (k_0 + k_\lambda + s_\lambda) I_\lambda(x, \theta) - s_\lambda \mathcal{J}_\lambda(x) - j_0(x) - j_\lambda(x), \tag{1}$$

where

$$\mathcal{J}_\lambda(x) = \frac{1}{4\pi}\int I_\lambda(x, \theta) d\omega. \tag{2}$$

Equation (1) is exact for all cases in which the "scattered" re-emission in the line takes place in the same wave-length interval as that in which it is absorbed, or microscopically speaking is applicable to transitions between two states, one of which, necessarily the lower, is infinitely sharp.

We now introduce Eddington's parameters

$$\eta_\lambda = (k_\lambda + s_\lambda)/k_0; \qquad \epsilon = k_\lambda/(k_\lambda + s_\lambda), \tag{3}$$

where η_λ is a function of wave-length across the line and depth in the atmosphere. The number ϵ, which from a microscopic point of view is the fraction of atoms excited to the upper state which are subsequently de-excited by a collision of the second kind, must from Weisskopf and Wigner's theory be a function of depth only. Thus for an infinitely sharp lower state the number of transitions upwards is proportional to $W(E)dE$ —the fraction of atoms in the upper state whose energies lie between E and $E + dE$; but the number of collisions of the second kind which atoms in the broadened upper state experience in this energy range is proportional to the product of $W(E)dE$ and the life-time. The life-time in the upper state is, however, independent of the position of the energy range, so that transitions up and collisions of the second kind both vary as $W(E)dE$, or ϵ is independent of wave-length.

If now we divide both sides of equation (1) by k_0, and introduce the parameters ϵ and η_λ, the right-hand side takes the form

$$(1 + \eta_\lambda) I_\lambda(x, \theta) - (1 - \epsilon)\eta_\lambda \mathcal{J}_\lambda(x) - \left(\frac{j_0(x)}{k_0} + \frac{j_\lambda(x)}{k_\lambda}\epsilon\eta_\lambda\right).$$

* P. ten Bruggencate, *Zs. f. Ap.*, 4, 159, 1932.

Following Strömgren *, though not defining our symbols in the same way, we may in the most general case put

$$j_\lambda(x)/k_\lambda = Qj_0(x)/k_0 = QB_\lambda(x). \tag{4}$$

Here Q, like ϵ and for the same reasons, is a number independent of wave-length; it varies with depth and tends to unity as the depth increases and the physical conditions more nearly approach local thermodynamical equilibrium. Again $B_\lambda(x)$ is not the Planckian specific intensity, but only a convenient symbol for $j_0(x)/k_0(x)$; it becomes the Planckian function only at those depths where local thermodynamical equilibrium holds.

Introduce now the optical depth for the wave-length λ in the continuous spectrum, defined by the equation

$$\tau_\lambda = \int_0^x k_0 \rho dx. \tag{5}$$

Further, it is convenient to divide the equation of transfer into an up-going part, where $I_\lambda(\tau_\lambda, \theta)$ is now the specific intensity of radiation directed towards the boundary $\tau_\lambda = 0$, making an angle θ ($0 \leqslant \theta \leqslant \pi/2$) with the negative direction of the τ_λ axis, and a down-going part, where $I'_\lambda(\tau_\lambda, \theta')$ is the specific intensity of the radiation directed towards the interior, making an angle θ' ($0 \leqslant \theta' \leqslant \pi/2$) with the positive direction of the τ_λ axis. With the new symbols defined in equations (3) and (4), but with no loss of exactness or generality, the equation of transfer now takes the form

$$\cos\theta \frac{dI_\lambda(\tau_\lambda, \theta)}{d\tau_\lambda} = (1+\eta_\lambda) I_\lambda(\tau_\lambda, \theta) - (1-\epsilon)\eta_\lambda \mathfrak{J}_\lambda(\tau_\lambda) - (1+\epsilon\eta_\lambda Q) B_\lambda(\tau_\lambda), \tag{6}$$

$$\cos\theta' \frac{dI'_\lambda(\tau_\lambda, \theta')}{d\tau_\lambda} = -(1+\eta_\lambda) I'_\lambda(\tau_\lambda, \theta') + (1-\epsilon)\eta_\lambda \mathfrak{J}_\lambda(\tau_\lambda) + (1+\epsilon\eta_\lambda Q) B_\lambda(\tau_\lambda), \tag{7}$$

$$\mathfrak{J}_\lambda(\tau_\lambda) = \tfrac{1}{2}\int_0^{\pi/2} I_\lambda(\tau_\lambda, \theta) \sin\theta d\theta + \tfrac{1}{2}\int_0^{\pi/2} I'_\lambda(\tau_\lambda, \theta') \sin\theta' d\theta'. \tag{8}$$

1·2. *The Integral Equations.*—To pass now to the observational model it is necessary to transform these last three equations into their integral form. For convenience in doing so we shall omit the subscript λ on the symbols η_λ and τ_λ, but nevertheless η still remains a function of λ and τ_λ, while τ remains a function of λ.

Multiply equation (6) through by the integrating factor $e^{-\sec\theta \int_t^\tau (1+\eta) dt}$, and (7) by the integrating factor $e^{\sec\theta \int_t^\tau (1+\eta) dt}$; on inserting the usual boundary conditions we obtain

$$I_\lambda(\tau, \theta) = \int_\tau^\infty K_\lambda(\tau) e^{-\sec\theta \int_\tau^t (1+\eta) dt} \sec\theta dt, \tag{9}$$

$$I'_\lambda(\tau, \theta') = \int_0^\tau K_\lambda(\tau) e^{-\sec\theta' \int_t^\tau (1+\eta) dt} \sec\theta' dt, \tag{10}$$

where

$$K_\lambda(\tau) = (1-\epsilon)\eta \mathfrak{J}_\lambda(\tau) + (1+\epsilon\eta Q) B_\lambda(\tau). \tag{11}$$

If now we insert (9) and (10) in (8), invert the order of integration and substitute $\mu = \sec\theta$, we obtain

$$\mathfrak{J}_\lambda(\tau) = \tfrac{1}{2}\int_0^\tau K_\lambda(t) \, . \, E_1\left(\int_t^\tau (1+\eta) dt\right) dt + \tfrac{1}{2}\int_\tau^\infty K_\lambda(t) \, . \, E_1\left(\int_\tau^t (1+\eta) dt\right) dt, \tag{12}$$

* B. Strömgren, *Zs. f. Ap.*, 10, 237, 1935.

where

$$E_1(x)=\int_1^\infty e^{-x\mu}\frac{d\mu}{\mu} \tag{13}$$

is the exponential integral. Equation (12) is essentially ten Bruggencate's * integral equation (24).

In order to find $K_\lambda(\tau)$ we pass in equation (9) to the specific intensity at $\tau=0$. The resulting surface brightness is given by

$$I_\lambda(0,\theta)=\int_0^\infty K_\lambda(t)\,.\,e^{-\sec\theta\int_0^t(1+\eta)dt}\sec\theta dt. \tag{14}$$

This is essentially ten Bruggencate's * integral equation (17). In the adjacent continuous spectrum, the surface brightness of which we may designate by $I_0(0,\theta)$, we know from (3) that $\eta=0$ and from (11) that $K_\lambda(\tau)=B_\lambda(\tau)$. With these substitutions (14) becomes

$$I_0(0,\theta)=\int_0^\infty B_\lambda(t)e^{-t\sec\theta}\sec\theta dt. \tag{15}$$

The integral equations (15), (14) and (12), which are exact, no simplifying approximations having been introduced at any stage, contain the solution of our problem. At any one wave-length we thus have three equations and five unknowns, namely $B_\lambda(\tau)$, $\mathcal{J}_\lambda(\tau)$, η, ϵ and Q. As first pointed out by ten Bruggencate * the solution is for a single wave-length indeterminate, but by introducing, as he showed, the known variation of the line absorption coefficient with wave-length it is possible to remove this indeterminacy.

1·3. *The Numerical Integration.*—In order to carry out the numerical integration of equations (15), (14) and (12) we first introduce the simply and directly observable quantities

$$I_0^*(0,\theta)=\frac{I_0(0,\theta)}{I_0(0,0)}\quad\text{and}\quad I_\lambda^*(0,\theta)=\frac{I_\lambda(0,\theta)}{I_0(0,\theta)}\cdot\frac{I_0(0,\theta)}{I_0(0,0)}=r_\lambda I_0^*(0,\theta). \tag{16}$$

If correspondingly we define

$$B_\lambda^*=B_\lambda/I_0(0,0);\qquad K_\lambda^*=K_\lambda/I_0(0,0);\qquad \mathcal{J}_\lambda^*=\mathcal{J}_\lambda/I_0(0,0), \tag{17}$$

then equations (15), (14) and (12) retain their present form unchanged. Now in equations (15) and (14) respectively, substitute

$$y=e^{-t\sec\theta}, \tag{18}$$

$$z=e^{-\sec\theta\int_0^t(1+\eta)dt}. \tag{19}$$

We thus finally obtain

$$I_0^*(0,\theta)=\int_0^1 B_\lambda^*(t)dy, \tag{20}$$

$$I_\lambda^*(0,\theta)=\int_0^1[K_\lambda^*(t)/(1+\eta)]dz, \tag{21}$$

$$\mathcal{J}_\lambda^*(\tau)=\tfrac{1}{2}\int_0^\tau K_\lambda^*(t)\,.\,E_1\left(\int_t^\tau(1+\eta)dt\right)dt+\tfrac{1}{2}\int_\tau^\infty K_\lambda^*(t)\,.\,E_1\left(\int_\tau^t(1+\eta)dt\right)dt, \tag{22}$$

where

$$K_\lambda^*(\tau)=(1-\epsilon)\eta\mathcal{J}_\lambda^*(\tau)+(1+\epsilon\eta Q)B_\lambda^*(\tau). \tag{23}$$

* P. ten Bruggencate, *Zs. f. Ap.*, 4, 159, 1932.

In order to carry out the integration divide the atmosphere with respect to τ into n layers * as follows:—

$$\left.\begin{array}{lll} \tau_1-\delta_1=0 & \text{to} & \tau_1+\delta_1 \\ \tau_2-\delta_2=\tau_1+\delta_1 & \text{to} & \tau_2+\delta_2 \\ \cdots\cdots\cdots & \cdots & \cdots \\ \tau_n-\delta_n=\tau_{n-1}+\delta_{n-1} & \text{to} & \tau_n+\delta_n \end{array}\right\}. \tag{24}$$

Within each of these layers, of thickness $2\delta_i$, the five functions of τ are assumed to take on characteristic and constant values, namely $B^*_\lambda(\tau_i)$, $\mathcal{J}^*_\lambda(\tau_i)$, η_i, ϵ_i, Q_i, with the consequence that $K^*_\lambda(\tau_i)$ is also a constant within the layer. Each of the integrals in equations (20), (21) and (22) is thus reduced to a summation. Let us consider each in turn.

1·31. Equation (20) takes the form

$$I^*_0(0,\theta)=\sum_{i=1}^{n} B^*_\lambda(\tau_i)\,.\,\Delta y. \tag{25}$$

This is a linear equation containing on the left-hand side the observed $I^*_0(0,\theta)$ at some particular value of θ, and on the right-hand side the sum of n terms. Each of these terms contains the unknown $B^*_\lambda(\tau_i)$, multiplied by the known Δy, a function of τ_i and θ, given for the ith layer from (18) by

$$\Delta y=e^{-(\tau_i-\delta_i)\sec\theta}-e^{-(\tau_i+\delta_i)\sec\theta}. \tag{26}$$

Thus n observed values of $I^*_0(0,\theta)$, obtained at n different points on the solar disc, will give us n simultaneous linear equations to solve for the n unknowns $B^*_\lambda(\tau_i)$.

It is important to note, especially since Raudenbusch †, in the discussion of his recent and accurate observations of darkening towards the limb, has failed to do so, that the division into n layers is a division according to τ, not according to y. If the latter division is made, the mid-point of the ith layer will from (18) occur at $-(\ln y_i)/\sec\theta$. Hence to each value of θ, and consequently in each equation (25), will occur a new set of n unknowns $B^*_\lambda(\tau)$, with the result that the n linear equations will contain n^2 unknowns. If, however, the division is made according to τ, as in (25), the number of unknowns and the number of equations is the same, so that a determinate solution is possible.

1·32. Equation (21) takes the form

$$I^*_\lambda(0,\theta)=\sum_{i=1}^{n}[K^*_\lambda(\tau_i)/(1+\eta_i)]\Delta z, \tag{27}$$

where from (19)

$$\Delta z=e^{-\sec\theta\int_0^{\tau_i-\delta_i}(1+\eta)dt}-e^{-\sec\theta\int_0^{\tau_i+\delta_i}(1+\eta)dt}, \tag{28}$$

and from (24)

$$\int_0^{\tau_i-\delta_i}(1+\eta)dt=2\sum_{j=1}^{i-1}(1+\eta_j)\delta_j; \qquad \int_0^{\tau_i+\delta_i}(1+\eta)dt=2\sum_{j=1}^{i}(1+\eta_j)\delta_j. \tag{29}$$

On the left-hand side of (27) we have the observed $I^*_\lambda(0,\theta)$, and on the right-hand side we have n terms, each a function of the two unknowns $K^*_\lambda(\tau_i)$ and η_i.

To make a start at the solution of the n equations (27), we must therefore assume a set of n values for the unknowns η_i. This enables us to calculate Δz from (28), with

* ten Bruggencate (*Zs. f. Ap.*, 4, 159, 1932, sec. 6) also proposes a method of numerical solution which depends upon dividing the atmosphere up into n layers according to τ. His method, however, is only applied to the approximate second order differential equation of the problem, not to the exact integral equations, and would not indeed have been either so elegant or so practicable if the correct boundary conditions for his layers had been used.

† H. Raudenbusch, *A.N.*, 266, 301, 1938.

the result that the n equations (27) contain now only the n unknowns $K_\lambda^*(\tau_i)/(1+\eta_i)$. The solution of the equations (27) carried out in just the same manner as the solution of the equations (25), will lead to values for the n unknowns $K_\lambda^*(\tau_i)/(1+\eta_i)$, but these are hypothetical values, conditioned by the values of η_i which had to be assumed to start the solution.

1·33. Equation (22) takes the form

$$\mathcal{J}_\lambda^*(\tau_i)=\tfrac{1}{2}\left[K_\lambda^*(\tau_i)\int_{\tau_i-\delta_i}^{\tau_i}E_1\left(\int_t^{\tau_i}(1+\eta)dt\right)dt+\sum_{j=1}^{i-1}K_\lambda^*(\tau_j)\int_{\tau_j-\delta_j}^{\tau_j+\delta_j}E_1\left(\int_t^{\tau_i}(1+\eta)dt\right)dt\right]$$
$$+\tfrac{1}{2}\left[K_\lambda^*(\tau_i)\int_{\tau_i}^{\tau_i+\delta_i}E_1\left(\int_{\tau_i}^{t}(1+\eta)dt\right)dt+\sum_{j=i+1}^{n}K_\lambda^*(\tau_j)\int_{\tau_j-\delta_j}^{\tau_j+\delta_j}E_1\left(\int_{\tau_i}^{t}(1+\eta)dt\right)dt\right]. \quad (30)$$

Each integral within the square brackets refers to a single layer, so that it is legitimate to take K_λ^* outside the sign of integration. The integral for the ith layer is, however, split into two parts, that referring to the upper half of this layer occurring in the first pair of square brackets, while that referring to the deeper half occurs in the second pair of square brackets.

Since hypothetical values for η_j and $K_\lambda^*(\tau_j)$ are known from § 1·32, the only problem arising in deriving a corresponding hypothetical value for $\mathcal{J}_\lambda^*(\tau_i)$ is a computational one. Consider first the two integrals referring to the ith layer, in which the integrand becomes infinite. It is readily seen that

$$\int_{\tau_i-\delta_i}^{\tau_i}E_1\left(\int_t^{\tau_i}(1+\eta)dt\right)dt+\int_{\tau_i}^{\tau_i+\delta_i}E_1\left(\int_{\tau_i}^{t}(1+\eta)dt\right)dt=\frac{2}{1+\eta_i}\int_0^{(1+\eta_i)\delta_i}E_1(x)dx$$
$$=\frac{2}{1+\eta_i}\left[1-\frac{\gamma(1+\eta_i)\delta_i}{2}+\frac{(1+\eta_i)^2\delta_i^2}{8}-\ldots-\int_{(1+\eta_i)\delta_i/2}^{1}\ln\frac{1}{x}dx+\int_{(1+\eta_i)\delta_i/2}^{(1+\eta_i)\delta_i}E_1(x)dx\right]. \quad (31)$$

The first step follows from the fact that the argument of the exponential integral may be written as $(1+\eta_i)(t-\tau_i)$, while the second is a consequence of expanding the exponential integral * in the form

$$E_1(x)=-\gamma+\ln\frac{1}{x}+\frac{x}{1\cdot1!}-\frac{x^2}{2\cdot2!}+\frac{x^3}{3\cdot3!}-\ldots, \quad (32)$$

where γ is Euler's constant. Equation (31) may be readily evaluated, and values for $\delta_i=\cdot04$ and for η_i varying between 0 and 15 will be found in Table IV, § 2·4.

In equation (30) we are now left with integrals for layers other than the ith, and since the integrands do not become infinite the integrals may be readily found from Simpson's one-third rule. For this purpose we need the value of the exponential integral at equal intervals of t over the range $\tau_j-\delta_j$ to $\tau_j+\delta_j$, and these can be found from Jahnke and Emde's Tables once the argument is known. It is readily seen that the argument for $\tau_j-\delta_j$ is given by

$$\int_{\tau_j-\delta_j}^{\tau_i}(1+\eta)dt=2\delta_j(1+\eta_j)+2\delta_{j+1}(1+\eta_{j+1})+\ldots+2\delta_{i-1}(1+\eta_{i-1})+\delta_i(1+\eta_i), \quad (33)$$

and similarly for any other value of t.

1·34. We now have for the assumed range of values of $\eta_i(1)$ (§ 1·32) hypothetical values of $K_1^*(\tau_i)$ and $\mathcal{J}_1^*(\tau_i)$, for some particular wave-length, say λ_1. Consider now a

* Edward's *Integral Calculus*, 2, 335, 1922.

second wave-length, λ_2. The corresponding values of $\eta_i(2)$ for it, assuming λ_1 and λ_2 to lie in the radiation-damping wing, will be

$$\eta_i(2) = \eta_i(1)\left(\frac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_0}\right)^2, \tag{34}$$

and with these we may now proceed as before to find hypothetical values of $K_2^*(\tau_i)$ and $\mathcal{J}_2^*(\tau_i)$. Repeating this process for a third wave-length, λ_3, we may now write equations (23) in the form

$$\left.\begin{aligned} K_1^*(\tau_i) &= (1-\eta_i(1))\mathcal{J}_1^*(\tau_i) + (1+\epsilon_i\eta_i(1)Q_i)B_\lambda^*(\tau_i) \\ K_2^*(\tau_i) &= (1-\eta_i(2))\mathcal{J}_2^*(\tau_i) + (1+\epsilon_i\eta_i(2)Q_i)B_\lambda^*(\tau_i) \\ K_3^*(\tau_i) &= (1-\eta_i(3))\mathcal{J}_3^*(\tau_i) + (1+\epsilon_i\eta_i(3)Q_i)B_\lambda^*(\tau_i) \end{aligned}\right\}. \tag{35}$$

In these equations for the ith layer, $\eta_i(2)$ and $\eta_i(3)$ may be expressed in terms of $\eta_i(1)$ by means of equation (34), while both ϵ_i and Q_i are from § 1·1 independent of wave-length. If now on solving equations (35) for the three unknowns η_i, ϵ_i, Q_i, the value of η_i so found agrees for each layer with that originally assumed, we then have the correct values for each of the unknowns. Otherwise, in somewhat the same way as ordinary numerical equations are solved by the iterative method, a new set of values of η_i must be assumed and the process repeated, if necessary again and again, until this condition is satisfied.

Section 2.—Application of the Observational Model to the Mg b Line, λ 5184

Since the whole purpose of an observational model is to supply numerical values of the march of the interesting variable or variables, it must now be shown that the method detailed in the foregoing section actually works when applied to a line profile observed across the solar disc. In order to make this test it has been assumed that $\epsilon = 0$, a physically interesting case which, while conserving the essential difficulties of the method, allows the work to be kept within reasonable compass. The effect of this assumption is that equations (35) reduce to a single equation for each layer, namely

$$K_\lambda^*(\tau_i) = \eta_i \mathcal{J}_\lambda^*(\tau_i) + B_\lambda^*(\tau_i), \tag{36}$$

with the consequence that, since only η_i has to be found, a single wave-length alone within the chosen absorption line need be considered.

Just as the solution of a numerical equation by any one of the iterative methods commences with an approximate value of the root, so in the present problem approximate values of η_i must first be found before any progress can be made. A satisfactory and simple method of finding such values is to solve for the case $\epsilon = 1$, and consequently the treatment of this case in § 2·3 precedes the treatment of the specific problem of $\epsilon = 0$ given in § 2·4.

2·1. *The Observational Material.*—The *Mg b* line, λ 5184, arising as it does from a metastable level, meets the requirements essential for an application of the method (§ 1·1). From Table IV of an earlier paper* values of $r(\theta)$ were plotted against λ for the three observed values of $\cos\theta$. A smooth curve was drawn through these points for each of the three profiles, and values of $r(\theta)$ read off for $\Delta\lambda = 0{\cdot}83$ A. away from the line centre. This particular distance was chosen, partly to avoid blended faint lines in the wings of

* H. H. Plaskett, *M.N.*, **91**, 870, 1931. About a year after numerical trials had commenced new and more accurate profiles for *Mg b* λ 5184 were given by Houtgast (*Zs. f. Ap.*, **16**, 43, 1938), and a new determination of limb darkening in the continuous spectrum was given by Raudenbusch (*A.N.*, **266**, 301, 1938). In order not to lose such gains as had already resulted from the preliminary trials, it was decided to carry on with the older observational material, the more so since as a test of the practicability of the method one set of observational material is as good as another.

λ 5184, and partly to be sufficiently far from the line centre to minimize distortion due to the finite resolving power of the spectroscope.

These observed values of $r(\theta)$, the mean of the red and the violet wing, can from (16) be converted into the required $I^*_\lambda(o, \theta)$ by multiplication by $I_0(o, \theta)/I_0(o, o)$ for the continuous spectrum. This latter quantity for λ 5184 was taken by interpolation from the graphically given results of Moll, Burger and van der Bilt.* The resulting values of

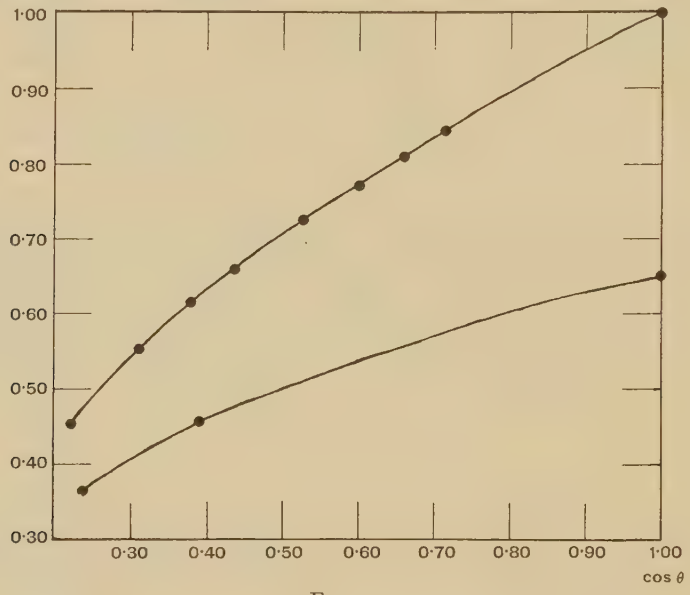

FIG. 1.

Upper Curve: Observed $I^*_0(o, \theta)$ *at* λ 5184.

Lower Curve: Observed $I^*_\lambda(o, \theta)$ *at* λ 5183·62 ±0·83.

$I^*_0(o, \theta)$ and of $I^*_\lambda(o, \theta)$, plotted against cos θ, are shown respectively in the upper and lower curves of fig. 1; for convenience of reference some numerical values, read from these curves, are given in Table I. From the estimates given in the original papers the error in $I^*_0(o, \theta)$ is of the order of ±1 per cent. and in $I^*_\lambda(o, \theta)$ of the order of ±5 per cent.

TABLE I

Values of $I^*_0(o, \theta)$ *and* $I^*_\lambda(o, \theta)$ *at* λ 5183·62 ±0·83

sec θ	1·00	1·50	2·00	2·50	3·00	3·50	4·00
$I^*_0(o, \theta)$	1·000	0·815	·706	·630	·573	·527	·485
$I^*_\lambda(o, \theta)$	0·650	·559	·498	·457	·423	·396	·372

2·2. *Observed Values of* $B^*_\lambda(\tau_i)$.—Since the function $B^*_\lambda(\tau_i)$ is independent of η and constant with respect to λ over the width of the absorption line, its value once found will be valid for all possible trial values of η. To find $B^*_\lambda(\tau_i)$ from the simultaneous equations (25), the solar atmosphere was divided into 17 layers, the limits of which are

* W. J. H. Moll, H. C. Burger and J. van der Bilt, *B.A.N.*, **3**, 83, 1925.

given in the first column of Table II, and values of Δy were computed from (26) for each layer and for each θ required. A table, similar to Table IV in an earlier paper *, was drawn up, and sets of values of $B^*_\lambda(\tau_i)$, subject to the condition that they varied smoothly with τ, were tried until one was found which reproduced the observed $I^*_0(0, \theta)$ in Table I. The thus found and adopted $B^*_\lambda(\tau_i)$, which reproduced the observed $I^*_0(0, \theta)$ within $\pm 0{\cdot}33$ per cent., that is well within the observational error, are given in the second column of Table II; though originally chosen only so as to reproduce $I^*_0(0, \theta)$ at the seven values of sec θ in Table I, it was subsequently found that they reproduced observation at ten intermediate values of sec θ with the same accuracy. This is presumably a result of choosing a set of values of $B^*_0(\tau_i)$ which varied smoothly with τ, and which therefore would be expected to yield a smoothly varying set of values of $I^*_0(0, \theta)$ with sec θ, in accordance with observation. In passing it may be noted that this same procedure of solving for 17 values of the unknown with only seven equations, but subjecting the unknown to the condition of a smooth variation with τ, was also used in all solutions of (27), with, as a consequence, a considerable saving in computation.

In addition to the adopted set of values of $B^*_\lambda(\tau_i)$ given in the second column of Table II, an independent determination was made which reproduced $I^*_0(0, \theta)$ within $\pm 0{\cdot}46$ per cent., also well within the observational error. This second determination

TABLE II

Values of $B^*_\lambda(\tau_i)$

τ_i $-\delta_i$	$+\delta_i$	Adopted $B^*_\lambda(\tau_i)$	Independent $B^*_\lambda(\tau_i)$	Error $(A-I)$	τ_i $-\delta_i$	$+\delta_i$	Adopted $B^*_\lambda(\tau_i)$	Independent $B^*_\lambda(\tau_i)$	Error $(A-I)$
				per cent.					per cent.
0·00	·08	0·233	0·232	+0·4	0·88	1·00	1·091	1·080	+1·0
·08	·16	·274	·272	+0·7	1·00	1·20	1·121	1·080	+3·7
·16	·24	·356	·354	+0·6	1·20	1·40	1·167	1·133	+2·9
·24	·32	·502	·499	+0·6	1·40	1·60	1·246	1·239	+0·6
·32	·40	·744	·740	+0·5	1·60	2·00	1·373	1·376	−0·2
·40	·52	0·945	0·965	−2·1	2·00	3·00	1·690	1·697	−0·4
·52	·64	1·046	1·070	−2·3	3·00	5·00	2·323	2·350	−1·2
·64	·76	1·066	1·080	−1·3	5·00	10·00	3·409	3·510	−3·0
·76	·88	1·081	1·080	+0·1					

is given in the third column of the table, while the fourth column contains the difference between the two determinations, expressed as a percentage of the adopted values in the second column. These differences, which lie between $-2{\cdot}9$ per cent. and $+3{\cdot}7$ per cent., arise from the fact that neither set of values of $B^*_\lambda(\tau_i)$ exactly satisfies the simultaneous equations, but only reproduces the observed $I^*_0(0, \theta)$ within, say, $\pm 0{\cdot}5$ per cent., itself half the estimated observational error. We must therefore anticipate that our adopted values of $B^*_\lambda(\tau_i)$ in the second column may have errors, originating from the observational errors in $I^*_0(0, \theta)$, amounting at any particular value of τ_i to as much as, say for safety, ± 4 per cent.

Dr. Busbridge †, in the discussion of her beautiful method of finding $B^*_\lambda(\tau_i)$, believes that even larger errors may be introduced by approximating to the solar atmosphere by a finite number of discrete layers. If this view is correct, it would be expected that a different division of the atmosphere into layers might lead to a different set of values of $B^*_\lambda(\tau_i)$. In an early stage of the present investigation the same division was tried as

* H. H. Plaskett, *M.N.*, **96**, 402, 1936. † I. W. Busbridge, *M.N.*, **101**, 26, 1941.

used in the first paper *, and the resulting values of $B^*_\lambda(\tau_i)$ are shown in fig. 2. It will be noted that, in spite of the much coarser division in the upper layers and the finer division in the deeper layers, this early solution fits the adopted solution very closely

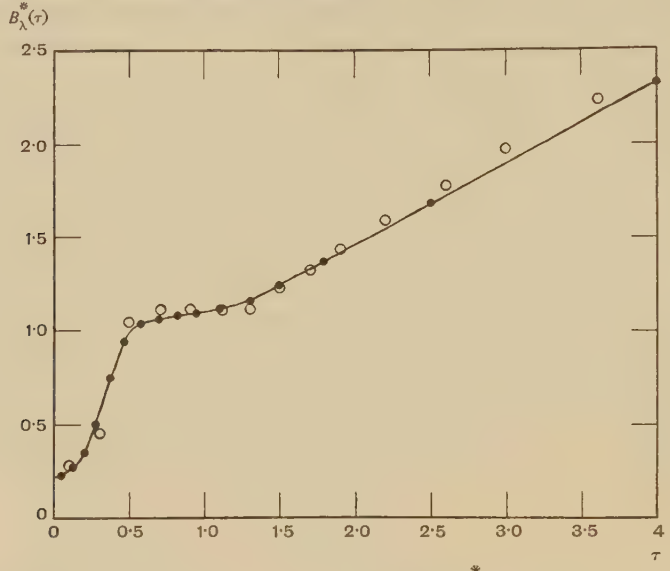

FIG. 2.—*Observed Values of* $B^*_\lambda(\tau)$.

Filled Circles: Adopted values from Table II.
Open Circles: Independent determination with a different division of atmosphere into n layers.

indeed, both solutions leading to an approximately isothermal region at the same place in the atmosphere. It is not improbable, therefore, that the absence of this region from Dr. Busbridge's solution arises from her assumption that

$$I^*_0(0, \theta) = (0{\cdot}92 \sec\theta + 0{\cdot}08)^{-0{\cdot}5}$$

is valid, not only over the observed region $0 \leqslant \sec\theta \leqslant 4$, but also in the almost inaccessible region $4 \leqslant \sec\theta \leqslant \infty$.

2·3. *Approximate Values of* η (*Case* $\epsilon = 1$).—When $\epsilon = Q = 1$, equation (23) takes the form

$$K^*_\lambda(\tau) = (1+\eta)B^*_\lambda(\tau). \tag{37}$$

Consequently (21) and (27) become

$$I^*_\lambda(0, \theta) = \int_0^1 B^*_\lambda(t)dz \tag{38}$$

$$= \sum_0^n B^*_\lambda(\tau_i) \,.\, \Delta z. \tag{39}$$

Since $B^*_\lambda(\tau_i)$ is already determined, the only unknowns in the equations (39) are the values of η_i which enter the Δz through (28). The solution of the simultaneous equations (39) provides, therefore, a method of finding approximate values of η with which to commence the solution of the physically interesting case $\epsilon = 0$.

In order to solve (39), an initial set of values of η_i is guessed, the corresponding Δz computed and multiplied by the known $B^*_\lambda(\tau_i)$ from the second column of Table II. The sum of the terms is then compared with the observed $I^*_\lambda(0, \theta)$. In the preliminary

* H. H. Plaskett, *M.N.*, **96**, 402, 1936.

stages considerable labour is saved if the comparison with $I^*_\lambda(0, \theta)$ is only made at sec $\theta = 1{\cdot}00$ and sec $\theta = 4{\cdot}00$. A few trials quickly serve to establish the nature of the variation—an increase or decrease of η_i with τ_i—and at the same time gives the rough absolute values required. The remaining trials are then made with comparisons at the seven values of $I^*_\lambda(0, \theta)$, the assumed η_i in each trial always, of course, being chosen so as to vary smoothly with τ.

The resulting solution of (39) is given in Table III. The first three columns contain τ_i, the adopted values of $1 + \eta_i$, and the values of $B^*_\lambda(\tau_i)$ from the second column of

TABLE III

Solution of $I^*_\lambda(0, \theta) = \Sigma B^*_\lambda(\tau_i) \,.\, \Delta z$

τ_i	$1 + \eta_i$	$B^*_\lambda(\tau_i)$	sec $\theta = 1{\cdot}00$		sec $\theta = 2{\cdot}00$		sec $\theta = 3{\cdot}00$		sec $\theta = 4{\cdot}00$	
			Δz	$B^*_\lambda \Delta z$	Δz	$B^*_\lambda \Delta z$	Δz	$B^*_\lambda \Delta z$	Δz	$B^*_\lambda \Delta z$
0·04	1·00	0·233	·0769	·0179	·1479	·0344	·2134	·0496	·2739	·0638
·12	1·13	·274	·0798	·0218	·1409	·0386	·1868	·0511	·2203	·0604
·20	1·39	·356	·0887	·0316	·1418	·0505	·1701	·0605	·1815	·0646
·28	3·90	·502	·2023	·1016	·2643	·1328	·2612	·1311	·2312	·1161
·36	8·30	·744	·2681	·1995	·2243	·1669	·1455	·1081	·0866	·0644
·46	9·71	·945	·1955	·1849	·0729	·0689	·0223	·0211	·0064	·0060
·58	10·80	1·046	·0644	·0674	·0073	·0076	·0007	·0007	·0001	·0001
·70	11·60	1·066	·0183	·0195	·0006	·0006				
·82	12·37	1·081	·0046	·0050						
·94	12·93	1·091	·0011	·0012						
1·10	13·62	1·121	·0003	·0003						
		$\Sigma B^*_\lambda \,.\, \Delta z =$		·6507		·5003		·4222		·3754
		$I^*_\lambda(0, \theta) =$		·650		·498		·423		·372

Table II. The succeeding four double columns contain Δz, and the product $B^*_\lambda(\tau_i) \,.\, \Delta z$ for each of four values of sec θ. At the foot of the columns are given for each value of sec θ the sums $\Sigma B^*_\lambda(\tau_i) \,.\, \Delta z$, and for comparison the observed $I^*_\lambda(0, \theta)$; it will be noted that the two agree well within 1 per cent. To economize in space the entries for the intermediate values of sec θ, namely 1·50, 2·50, 3·50, are not given, but the corresponding sums are, as can be readily verified, 0·5611, 0·4558 and 0·3965 respectively; these too agree within 1 per cent. with the observed $I^*_\lambda(0, \theta)$ as given in Table I. It will be noted that only 11 of the 17 layers have been used in the solution. This is a result of the rapid increase in η_i which the observations require, and which has the effect that Δz becomes less than 0·00005 for the 12th and deeper layers, so that the product $B^*_\lambda(\tau_i) \,.\, \Delta z$ for these layers becomes negligibly small.

2·4. *Observed Values of* η *for* $\epsilon = 0$.—Having derived observed values of B^*_λ in § 2·2, and a first approximation to the values of η in § 2·3, we are now in a position to solve for η in the case $\epsilon = 0$. The procedure, it may be recalled, is to assume a set of values of η_i for the 17 discrete layers, from these to compute Δz from (28) and (29), then to solve the simultaneous equations (27) for $K^*_\lambda/(1 + \eta)$, and finally from (30) to find $\mathcal{J}^*_\lambda$. This sequence of operations is termed a trial, and its success or failure is determined by whether the so-found values satisfy equations (36) for each of the 17 layers.

In equation (36) each of the quantities B_λ^*, K_λ^*, $\mathcal{J}_\lambda^*$ is subject to an observational uncertainty due to the observational error in $I_0^*(0, \theta)$ and $I_\lambda^*(0, \theta)$. The problem of satisfying (36) is therefore analogous to solving a numerical equation, such as

$$f(x) = C \tan x + De^{-x} = 0,$$

where C and D have independent observational errors. For any value of x the corresponding error in $f(x)$ is

$$\Delta f = \sqrt{(\Delta C \,.\, \tan x)^2 + (\Delta D \,.\, e^{-x})^2}.$$

A simple method of solving this numerical equation would be to test each trial root, x, and see whether it makes $f(x) \leqslant |\Delta f|$, the error Δf being calculated as if x were errorless. If it is desired to find the error in the root, which ultimately satisfies this criterion, the procedure would be to make further trials for the extreme values of x which make $f(x) \leqslant |\Delta f|$, and the required error would be the difference of these extremes.

Similarly in satisfying (36), which we may write in the form

$$f(\eta) = K_\lambda^* - (B_\lambda^* + \eta \mathcal{J}_\lambda^*), \tag{40}$$

we need seek only those values η which make $f(\eta) \leqslant |\Delta f|$, where

$$\Delta f = \sqrt{(\Delta K_\lambda^*)^2 + (\Delta B_\lambda^*)^2 + (\eta \Delta \mathcal{J}_\lambda^*)^2}. \tag{41}$$

In the derivation of (41) η is of course treated as errorless, while the errors in B_λ^*, K_λ^*, $\mathcal{J}_\lambda^*$ are assumed, as is the case, to be independent.

2·41. *The Result of Trial I.*—While from physical considerations it is clear that the approximate values of η, derived in § 2·3, must be an upper limit to the true values, it is convenient to start our solution with this particular approximation. This assumed set of values, which we shall denote as Trial I, leads by means of equations (27) to a set of values of $K_\lambda^*/(1+\eta)$. By virtue of the method which led to the Trial I values of η, the resulting values of $K_\lambda^*/(1+\eta)$ are simply B_λ^*; this convenient "accident", peculiar to Trial I, enables us to omit this stage in the computations, and leaves only $\mathcal{J}_\lambda^*$ to be found.

2·411. *Computation of* $\mathcal{J}_\lambda^*(\tau_i)$.—The computation of that part of $\mathcal{J}_\lambda^*(\tau_i)$ given by (31) can be carried out once and for all, and the results for the case $\delta_i = 0{\cdot}04$ and for different values of η are given in Table IV. The two integrals in (31) are evaluated by Simpson's one-third rule, and some care is needed that the interval for the integral involving $ln\,1/x$ is sufficiently finely subdivided, especially for small values of η. There is no necessity

TABLE IV

Values of $\dfrac{2}{1+\eta_i}\displaystyle\int_0^{0{\cdot}04(1+\eta_i)} E_1(x)dx$

$1+\eta_i$	$\frac{2}{1+\eta_i}\int$	$1+\eta_i$	$\frac{2}{1+\eta_i}\int$	$1+\eta_i$	$\frac{2}{1+\eta_i}\int$
1·00	0·2928	2·10	0·2351	7·00	0·1463
1·10	·2854	2·30	·2283	9·00	·1291
1·30	·2721	2·50	·2219	11·00	·1158
1·50	·2612	2·70	·2162	15·00	·0964
1·70	·2511	3·00	·2082		
1·90	·2428	5·00	·1704		

to reproduce the details of the calculation, and it suffices to say that the results in Table IV are accurate to within less than one-tenth of 1 per cent.

TABLE V

Evaluation of $\mathcal{J}_\lambda^*(0.36)$ *for Trial I*

t	$1+\eta_i$	$\int_t^{0.36}(1+\eta_i)dt$	E_1	Sum	$\int_{\tau_j-\delta_j}^{\tau_j+\delta_j}$	K_λ^*	$K_\lambda^*\int_{\tau_j-\delta_j}^{\tau_j+\delta_j}$
0·00		0·925	0·246				
·04	1·00	·885	·266	1·596	0·0213	0·233	0·0050
·08		·845	·286				
·08		·845	·286				
·12	1·13	·800	·310	1·864	·0249	·310	·0077
·16		·755	·338				
·16		·755	·338				
·18		·727	·358				
·20	1·39	·700	·376	4·527	·0302	0·495	·0150
·22		·672	·397				
·24		·644	·417				
·24		·644	·417				
·26		·567	·484				
·28	3·90	·488	·576	7·079	·0472	1·958	·0924
·30		·410	·686				
·32		·332	·830				
·36	8·30				·1344	6·17	·8300
·40		·332	·830				
·43		·624	·433				
·46	9·71	·915	·252	3·786	·0379	9·18	·3478
·49		1·21	·155				
·52		1·50	·100				
0·52		1·50	·1000				
·55		1·82	·0621				
·58	10·80	2·15	·0399	·5483	·00548	11·30	·0619
·61		2·47	·0258				
·64		2·80	·0169				
·64		2·80	·0169				
·70	11·60	3·50	·0070	·0479	·000958	12·37	·0118
·76		4·19	·0030				
·76		4·19	·00300				
·82	12·37	4·93	·00124	·00850	·0001700	13·38	·0023
·88		5·67	·00054				
·88		5·67	·00054				
0·94	12·93	6·45	·00022	·00153	·0000306	14·12	·0004
1·00		7·22	·00011				
1·00		7·22	·00011				
1·10	13·62	8·58	·00002	·00019	·0000063	15·29	·0001
1·20		9·94	·00000				

$\mathcal{J}_\lambda^*(0.36) = \frac{1}{2}(1.3744)$

The remainder of the computation for $\mathcal{J}_\lambda^*(\tau_i)$ is illustrated in Table V for the case $\tau_i = 0{\cdot}36$ of Trial I. The first column contains t, arranged in groups according to layers, the second column the values of $1+\eta_i$ appropriate to each layer for Trial I, while the third column contains

$$\int_t^{0{\cdot}36} (1+\eta)dt,$$

the argument of the exponential integral. The mode of computing this argument is indicated in (33), and the details of the calculation may be readily checked in the table. The fourth column contains the value of the exponential integral, the fifth these values summed according to Simpson's one-third rule, and the sixth the value of the integral

$$\int_{\tau_j-\delta_j}^{\tau_j+\delta_j} E_1\left(\int_t^{0{\cdot}36} (1+\eta)dt\right)dt,$$

found by multiplying this sum by one-third the interval. The interval has been chosen sufficiently small to keep the computational errors to the order of one-tenth of 1 per cent. The value of the integral for the layer $\tau_i = 0{\cdot}36$ has of course been taken from a curve drawn through the values given in Table IV, and consequently for this layer there are no entries in the second, third and fourth columns. The last two columns contain $K_\lambda^*(\tau_i) = (1+\eta_i)B_\lambda^*(\tau_i)$ (equation (37)), and the product of $K_\lambda^*(\tau_i)$ and the integral in the sixth column. From equation (30) it follows that $\mathcal{J}_\lambda^*(0{\cdot}36)$ is one-half the sum of the entries in this last column.

2·412. *Values of $f(\eta)$ for Trial I.*—In a precisely similar way the values of $\mathcal{J}_\lambda^*$ are found for other values of τ_i. These are given in the third column of Table VI, the first two columns of which contain τ_i and the Trial I values of η_i. This table is carried only

TABLE VI

Values of $f(\eta) = K_\lambda^* - (B_\lambda^* + \eta\mathcal{J}_\lambda^*)$ for Trial I

τ_i	η_i	$\mathcal{J}_\lambda^*$	$\eta\mathcal{J}_\lambda^*$	B_λ^*	$B_\lambda^* + \eta\mathcal{J}_\lambda^*$	K_λ^*	$f(\eta)$
0·04	0·00	0·266	0·000	0·233	0·233 ± ·009	0·233 ± ·009	± 0·000 ± ·013
·12	·13	·314	·041	·274	·315 ·011	·310 ·012	− ·005 ·016
·20	·39	·371	·145	·356	0·501 ·015	0·495 ·020	− ·006 ·025
·28	2·90	·482	1·399	·502	1·901 ·059	1·958 ·078	+ ·057 ·098
·36	7·30	·687	5·01	·744	5·75 ·20	6·17 ·25	+ ·42 ·32
·46	8·71	·906	7·90	·945	8·84 ·32	9·18 ·37	+ ·34 ·49
·58	9·80	1·028	10·07	1·046	11·12 ·40	11·30 ·45	+ ·18 ·60
·70	10·60	1·060	11·22	1·066	12·29 ·45	12·37 ·49	+ ·08 ·66
·82	11·37	1·075	12·21	1·081	13·29 ·49	13·38 ·53	+ ·09 ·72

as far as $\tau_i = 0{\cdot}82$, since computation of $\mathcal{J}_\lambda^*$ for deeper layers cannot be effected with the necessary computational accuracy in the absence of values of K_λ^* and η for $\tau_i \geqslant 1{\cdot}30$ (see Table III). The fourth and fifth columns of Table VI contain the values of $\eta\mathcal{J}_\lambda^*$ and of B_λ^*, the latter taken from the second column of Table II, while the sixth and seventh columns contain respectively the sum $B_\lambda^* + \eta\mathcal{J}_\lambda^*$ and the quantity K_λ^*, each with their observational errors. The amount of these errors is found from the following considerations. The error in B_λ^*, resulting from a solution of the simultaneous equations (25) which satisfy $I_\lambda^*(0, \theta)$ within ±0·5 per cent., is found in § 2·2 to be ±4 per cent. Since in the general case the values of $K_\lambda^*/(1+\eta)$ are found from the solution of the

analogous simultaneous equations (27), which satisfy $I^*_\lambda(0, \theta)$ within $\pm 0{\cdot}5$ per cent.*, it may be assumed that the error in $K^*_\lambda/(1+\eta)$ is also ± 4 per cent. The error in K^*_λ will be identical with this, since η is to be treated as errorless. Finally, apart from negligible computational errors (less than 0·1 per cent.), the error in $\mathcal{J}^*_\lambda$ arises wholly from observational errors in K^*_λ, and consequently $\mathcal{J}^*_\lambda$ will likewise have an error of ± 4 per cent.

The final column of Table VI contains $f(\eta)$ with its observational error, computed from the individual terms and their errors in accordance with equations (40) and (41). From the discussion in § 2·4 it follows that our Trial I values of η will be correct if, and only if, $f(\eta)$ is zero within its observational error for all values of τ_i. It will at once be seen that this condition is not satisfied at $\tau_i = 0{\cdot}36$, for which $f(\eta) = 0{\cdot}42 \pm 0{\cdot}32$.

2·42. *Results of Subsequent Trials.*—Since, as already indicated, the Trial I values of η must be an upper limit to the true values, it is clear that smaller values should be experimented with in Trial II. This is also confirmed by experience, which has shown that positive values of $f(\eta)$ for a given layer can generally be made zero or negative by decreasing the values of η in neighbouring layers. The second and third columns of Table VII contain the values of η chosen for the second trial, and the resulting values of $f(\eta)$ with its observational error, determined as above from equations (40) and (41).

TABLE VII

Trial and Final Values of η

τ_i	Trial II		Trial III		Trial VII		Trial XI	
	η	$f(\eta)$	η	$f(\eta)$	η	$f(\eta)$	η	$f(\eta)$
0·04	0·00	$-{\cdot}004 \pm {\cdot}013$	0·00	$-{\cdot}003 \pm {\cdot}013$	0·05	$-{\cdot}021 \pm {\cdot}013$	0·00	$-{\cdot}011 \pm {\cdot}013$
·12	·03	$-{\cdot}008$ ·016	·08	$-{\cdot}028$ ·016	·10	$-{\cdot}012$ ·016	·02	$-{\cdot}015$ ·016
·20	·09	$-{\cdot}016$ ·021	·24	$-{\cdot}008$ ·022	·32	$+{\cdot}005$ ·024	0·08	$-{\cdot}009$ ·020
·28	0·45	$-{\cdot}068$ ·033	0·98	$+{\cdot}003$ ·045	0·92	$+{\cdot}018$ ·045	1·00	$-{\cdot}013$ ·045
·36	1·61	$-{\cdot}139$ ·072	3·00	$+{\cdot}032$ ·122	2·30	$+{\cdot}025$ ·100	3·00	$-{\cdot}03$ ·11
·46	4·52	$-{\cdot}01$ ·21	5·60	$-{\cdot}05$ ·27	3·68	$\pm{\cdot}00$ ·18	5·35	$+{\cdot}12$ ·26
·58	6·20	$+{\cdot}24$ ·35	7·10	$+{\cdot}20$ ·40	4·45	$\pm{\cdot}00$ ·25	7·10	$+{\cdot}16$ ·40
·70	6·49	$+{\cdot}27$ ·39	8·10	$+{\cdot}14$ ·52	4·88	$+{\cdot}06$ ·30	8·10	$+{\cdot}13$ ·50
·82	6·60	$+{\cdot}01$ ·42	8·80	$+{\cdot}06$ ·59	5·15	$+{\cdot}07$ ·33	8·80	$+{\cdot}12$ ·59
·94	6·66	$-{\cdot}05$ ·44			5·32	$+{\cdot}04$ ·36	9·30	$+{\cdot}07$ ·64

It will be seen that the Trial II values of η have given us a lower limit, since $f(\eta)$ is now negative, outside the observational error, for $\tau_i = 0{\cdot}28$ and 0·36. In the fourth and fifth columns of this table appear the intermediate values of η chosen for the third trial with the resulting values of $f(\eta)$ and its observational error. This is a quite promising trial, and apart from a negative $f(\eta)$ at $\tau_i = 0{\cdot}12$, all values are zero within the observational error.

At this stage, unfortunately some time was wasted in trying to reduce the absolute values of $f(\eta)$ without paying due attention to its observational error, and Trial VII, given in the sixth and seventh columns of the table, shows that some measure of success

* It is true, of course, that the actual observational error in $I^*_\lambda(0, \theta)$ is $\pm$ 5 per cent. (see § 2·1), but it is legitimate, in order to make the present numerical solution typical of one carried out with more ideal observational material, to treat the observational error in $I^*_\lambda(0, \theta)$ as being of the same order of magnitude as that in $I^*_0(0, \theta)$.

was attained in this unimportant respect. This trial, however, like its four predecessors, shows the negative value of $f(\eta)$, outside the observational error, for $\tau_i = 0{\cdot}04$ or $0{\cdot}12$. In order to remove this negative value various somewhat drastic variations of η were tried, and it was ultimately found that this could be done, contrary to the rule given at the beginning of this subsection, by decreasing η for the three first layers. With this as a clue, the successful Trial XI was carried through, the values of η and the corresponding values of $f(\eta)$ being given in the last two columns of Table VII. It will at once be seen that all values of $f(\eta)$ are zero within their observational error, from

FIG. 3.—*Observed Values of* η.

For $\epsilon = 0$ (*filled circles*) *and for* $\epsilon = 1$ (*open circles*).

which it follows that the Trial XI values of η are those which satisfy the observed variation of the *Mg b* line λ 5184 across the disc of the Sun.

2·43. The error in η, consequent upon the observational uncertainty in $I^*_\lambda(0, \theta)$, is not directly given by the method of analysis. This is not, however, an insuperable difficulty, since it would be possible, just as in § 2·2, to make one or two additional trials, and so find the maximum variation in η for which $f(\eta)$ could be made zero within the observational error.

2·5. *Discussion of the Results.*—The values of η for the case $\epsilon = 0$ (Trial XI, Table VII) and for the case $\epsilon = 1$ (Table III) are shown graphically as functions of τ in fig. 3. These values are correct within their observational error if the observed values of $I^*_\lambda(0, \theta)$ are correct within $\pm 0{\cdot}5$ per cent. On account of the actual ± 5 per cent. uncertainty in this latter quantity (§ 2·1), the resulting values of η are not entitled to much confidence as representative of the actual physical conditions in the Sun. Two features of the variation are perhaps, however, worthy of comment, namely the increase of η with depth, and the low values near the boundary $\tau = 0$. The first of these is presumably the result

of the fact that the metastable lower state for the *Mg b* line λ 5184 has an excitation potential of as much as 2·7 volts, with the consequence that the population of neutral atoms in that state increases with increasing temperature, that is with increasing depth. A resonance line, on the other hand, would presumably show a decrease of η with depth. The second feature, the vanishingly small values of η near $\tau = 0$, is inexplicable unless the temperature is low near the boundary, but it does suggest that the absorption line is formed below the surface layers. The consequence of such a mode of formation is that the continuous spectrum originating in these surface layers will tend to fill in the centre of the line, and this goes some way towards accounting for the characteristic non-zero central intensity shown by this line.

Summary and Conclusions

1. The purpose of the investigation has been to set up an *observational* model, as contrasted with the more usual *mathematical* model, for absorption lines.

2. The fixed hypothetical element of both types of models is the equation of transfer. An exact form of this, applicable to all absorption lines having an infinitely sharp lower state, and which makes no assumption about local thermodynamic equilibrium, is given in equations (6), (7) and (8).

3. The corresponding integral equations required for the observational model are derived and given in a form suitable for the analysis of observations in equations (20) to (23) inclusive.

4. By a division of the solar atmosphere into n layers, the solution of two of these integral equations is reduced to the solution of n simultaneous equations, while the third integral equation is reduced to a summation of easily found individual terms.

5. This method of analysis is applied to the observed variation of the *Mg b* line λ 5184 across the solar disc, and the observed values of η as a function of depth are derived for the case $\epsilon = 0$. These values are shown graphically in fig. 3.

6. To bring the numerical work to a successful conclusion it has been found essential first to solve for the simple case $\epsilon = 1$. This delimits the variation in η. Secondly, it has been necessary to keep the computational errors in $\mathcal{J}_\lambda^*$ within 0·1 per cent.; and finally the solution has only been effected by bearing in mind that $f(\eta)$ is to be made zero, for all values of τ_i, only within the observational error of $f(\eta)$ itself—any further refinement being meaningless.

The observational model for absorption lines yields directly an observed variation of the line absorption coefficient with depth in the solar atmosphere. Combined with the brightness temperature and the continuous absorption coefficient as a function of depth, which are given by the observational model for the continuous spectrum, it should ultimately be possible to set up a framework of observed temperatures and pressures with depth in that atmosphere. For this to be effected with any precision, there is need for observations of the highest accuracy on limb variations both of continuous spectrum and of lines originating from normal and metastable states. At the same time there is clearly room for investigation of improved mathematical methods for effecting the necessary analysis of the observations. In this latter respect the work of Dr. Busbridge on the continuous spectrum, and some recent, as yet unpublished, work of Mr. D. S. Evans on both continuous and line spectra, are of considerable interest and promise.

The present investigation had its origin in some lectures which I was invited to give at the University of London in the spring of 1937, and an account of the theory as detailed in the first section and the result of a single trial were given at Dr. Woolley's colloquium in Cambridge in November of that year. Following this, seventeen further trials were carried through without success, and the first real progress was not made until the spring of 1939 when the case for $\epsilon = 1$ was completed. In spite of set-backs due to insufficient

accuracy in the computation of $\mathcal{J}_\lambda^*$, Trial III was finished by the middle of August. Since that time work on the problem has necessarily been a spare-time occupation carried on under a variety of circumstances. I should like to acknowledge the encouragement which I have received throughout the course of the investigation from the interest of Professor Milne, and at a later stage, during his visit to Oxford, of Dr. Theodore Dunham, jun. I am also indebted to Dr. A. D. Thackeray for helpful suggestions with regard to the exposition of the final section.

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THE INVERSE PROBLEM OF STELLAR ABSORPTION LINE FORMATION

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(Communicated by the Director, University Observatory, Oxford)

1. The direct problem of stellar absorption line formation consists in the deduction from the known characteristics of a stellar atmosphere of the contour of a Fraunhofer line at all points on the disc of a star. The inverse problem is the deduction of the characteristics of the atmosphere from observations of the continuous spectrum and line contour across the disc.

This paper contains a formal solution in series of the inverse problem under somewhat restricted conditions. It is unfortunately not possible to adapt it for numerical calculation for reasons which will be explained, but it does demonstrate that the solution under the conditions postulated is a unique one, and hence shows what information is deducible from observations of centre to limb variations of line contours.

The problem is as follows :—A Fraunhofer line is formed in a stellar atmosphere under conditions such that the proportions of atoms de-excited by super-elastic collisions and through uncompensated cycles of atomic transitions are known constants (ϵ and δ respectively). It is required to determine from observations of the centre to limb variations of the contour, the value of η at each wave-length for all optical depths. It is assumed that observations of the neighbouring continuous spectrum are available, and that, apart from the de-excitation through uncompensated cycles, the scattering processes are purely isochromatic.

The equation of transfer for the formation of the line is *

$$\cos\theta\frac{d\mathcal{J}_\nu'(\theta)}{d\tau}=(1+\eta)\mathcal{J}_\nu'(\theta)-(1-\epsilon-\delta)\eta\mathcal{J}_\nu'-\epsilon\eta\mathcal{J}_T-\mathcal{J}_T, \tag{1}$$

where

$$\mathcal{J}_\nu'=\frac{1}{4\pi}\int\mathcal{J}_\nu'(\theta)d\omega, \tag{2}$$

and we assume

$$\mathcal{J}_T=\tau^\alpha\sum_0^\infty a_r\tau^r, \tag{3}$$

in which $0<\alpha<1$. The continuous spectrum is given by equation (1) with η zero, and we readily find † that at the boundary of the star

$$I(\theta)=\cos^\alpha\theta\sum_0^\infty a_r\,.\,\Gamma(\alpha+r+1)\cos^r\theta. \tag{4}$$

We now assume

$$\mathcal{J}_\nu'(\theta)=\cos^\beta\theta\sum_0^\infty\sum_0^\infty\mathcal{J}_{rs}\tau^r\cos^s\theta, \tag{5}$$

where $0<\beta<1$ and there is no fractional power of τ, since $\mathcal{J}_\nu'(\theta)$ cannot vanish at the stellar surface.

* Eddington, *M.N.*, **89**, 620, 1929.

† See Minnaert, *Zs. f. Ap.*, **12**, 313, 1936.

$r(\theta)$, the observed residual intensity at a point θ on the disc, is then given by

$$r(\theta) = \frac{\cos^{\beta}\theta \sum_{0}^{\infty} \mathfrak{J}_{0s} \cos^{s}\theta}{\cos^{\alpha}\theta \sum_{0}^{\infty} a_r \cos^{r}\theta \,.\, \Gamma(\alpha + r + 1)}$$

$$= \cos^{r}\theta \sum_{0}^{\infty} p_s \cos^{s}\theta, \qquad \text{say, where } \alpha + r = \beta. \tag{6}$$

We now write η in the form

$$\eta = \tau^{\zeta} \sum_{0}^{\infty} \eta_r \tau^{r}, \tag{7}$$

where ζ is a positive fraction, and then we have by (1),

$$1 + \eta = \frac{\dfrac{\partial}{\partial \cos\theta}\left\{\cos\theta \dfrac{d\mathfrak{J}_{\nu}'(\theta)}{d\tau}\right\}}{\dfrac{\partial}{\partial \cos\theta}\{\mathfrak{J}_{\nu}'(\theta)\}}. \tag{8}$$

Now the left-hand side of (8) is independent of θ and the right-hand side is the ratio of two ordinary power series in τ. Hence the left-hand side cannot contain any fractional power of τ either and hence ζ is zero.

Further, β must be zero also, for if our series for $\mathfrak{J}_{\nu}'(\theta)$ is substituted in (1) and we equate powers of $\cos\theta$, the first term on the right-hand side gives us an uncompensated term in $\cos^{\beta}\theta$, and we must therefore either have $(1+\eta)$ always zero, or β zero, in which case when we equate coefficients this term is taken in with the other terms in τ only on the right-hand side of (1).

It therefore follows that η, $\mathfrak{J}_{\nu}'(\theta)$ (and hence $\mathfrak{J}_{\nu}'$) contain no terms in fractional powers of τ or $\cos\theta$, and hence a glance at (1) shows that $\mathfrak{J}_{T}$ cannot do so either, *i.e.* α is zero.

We may now proceed to equate powers of $\cos\theta$ and of τ in (8) and derive the equations

$$n\mathfrak{J}_{n,\,r} = (1+\eta_0)\mathfrak{J}_{n-1,\,r+1} + \eta_1 \mathfrak{J}_{n-2,\,r+1} + \ldots \tag{9}$$

Hence the matrix of coefficients $\mathfrak{J}_{rs}$ may be written (equation (10))

$\mathfrak{J}_{00}$	$\mathfrak{J}_{01}$	$\mathfrak{J}_{02}$	$\mathfrak{J}_{03}$
$(1+\eta_0)\mathfrak{J}_{01}$	$(1+\eta_0)\mathfrak{J}_{02}$	$(1+\eta_0)\mathfrak{J}_{03}$	$(1+\eta_0)\mathfrak{J}_{04}$
$\frac{1}{2}\{(1+\eta_0)^2\mathfrak{J}_{02} + \eta_1\mathfrak{J}_{01}\}$	$\frac{1}{2}\{(1+\eta_0)^2\mathfrak{J}_{03} + \eta_1\mathfrak{J}_{02}\}$	$\frac{1}{2}\{(1+\eta_0)^2\mathfrak{J}_{04} + \eta_1\mathfrak{J}_{03}\}$	$\frac{1}{2}\{(1+\eta_0)^2\mathfrak{J}_{05} + \eta_1\mathfrak{J}_{04}\}$
$\frac{1}{6}\{(1+\eta_0)^3\mathfrak{J}_{03} + 3\eta_1(1+\eta_0)\mathfrak{J}_{02} + 2\eta_2\mathfrak{J}_{01}\}$	. . .	. . .	. . .
. . .	. . .	. . .	. . .

i.e. the first n rows may be expressed in terms of the first $n-1$ η_r's and the $\mathfrak{J}_{0s}$.

Finally the expression for $\mathfrak{J}_{\nu}'$ is found to be

$$\mathfrak{J}_{\nu}' = \frac{1}{4\pi}\int \sum_{0}^{\infty}\sum_{0}^{\infty} \mathfrak{J}_{rs}\tau^{r}\cos^{s}\theta = \sum_{0}^{\infty}\sum_{0}^{\infty} \frac{\mathfrak{J}_{r,\,2p}}{2p+1}\tau^{r}, \tag{11}$$

and substituting this, and the series (4), (5) and (7) in (1), and equating coefficients of powers of τ, we find the equations

$$0 = \mathfrak{J}_{r0} + \sum_{s=0}^{r} \eta_{r-s}\mathfrak{J}_{s0} - (1-\epsilon-\delta)\sum_{s=0}^{r}\sum_{p=0}^{\infty} \eta_{r-s}\frac{\mathfrak{J}_{s,\,2p}}{2p+1} - \epsilon\sum_{s=0}^{r}\eta_{r-s}a_s - a_r, \tag{12}$$

so that, for instance,

$$\eta_0 = \frac{\mathcal{J}_{00} - a_0}{(1-\epsilon-\delta)\left\{\mathcal{J}_{00} + \frac{\mathcal{J}_{02}}{3} + \frac{\mathcal{J}_{04}}{5} + \ldots\right\} + \epsilon a_0 - \mathcal{J}_{00}}, \tag{13}$$

and in this way the coefficients of the series for η can be determined successively in terms of the elements of the matrix $\mathcal{J}_{rs}$, *i.e.* by (11) in terms of previously determined η_r's and the elements $\mathcal{J}_{0s}$.

Now our hypothetical observational data include curves of observations of $I(\theta)$ and $I(\theta).r(\theta)$, and these have, at the point $\cos\theta = 0$, a perfectly definite gradient, first curvature and derivatives of all orders, so that in principle they are represented by unique power series, which by (4) and (6) determine the a_r and the $\mathcal{J}_{0s}$ uniquely, and hence determine η uniquely.

It is obvious that this programme cannot be carried out in practice, both because of the impossibility of carrying the observations to the extreme limb, and of the inaccuracy of determination of more than the first two or three differential coefficients at any point on a curve given only graphically.

2. *Convergence.*—A complete discussion of the conditions under which a solution of the inverse problem exists would be a very complicated matter, and would include cases which are physically impossible, such as negative values for η and distributions having various kinds of discontinuity. Our objective is the more limited one of showing that if a physically reasonable solution has been found, then there is no second solution of the same type which is consistent with the same observational data. This will be the case if the analysis of the preceding section is legitimate, for the equations determine η uniquely as a function of optical depth.

It will be assumed, therefore, that a solution for η has been found by some means and that this solution is valid to an optical depth greater than unity. Hence $\sum_r |\eta_r|$ converges, and so after some finite point $|\eta_r|$ is a monotonic decreasing sequence (broad sense). Further, $\Sigma\mathcal{J}_{0s}\cos^s\theta$ converges, for $0 \leqslant \cos\theta \leqslant 1$, so that after some finite point $|\mathcal{J}_{0s}|$ is a decreasing sequence in the broad sense. As a first step we modify these two sequences so as to obtain two decreasing sequences ρ_r and P_{0s}, differing by only a finite number of terms from $|\eta_r|$ and $|\mathcal{J}_{0s}|$, and having the following properties:—

$$\left.\begin{array}{l} \text{(i) } \rho_0 > |1+\eta_0|; \quad \text{(ii) } \rho_r \geqslant |\eta_r|; \quad \text{(iii) } P_{0s} \geqslant |\mathcal{J}_{0s}|; \\ \text{(iv) } \sum_r \rho_r \text{ converges}; \quad \text{(v) } P_{0s} \to 0 \text{ as } s \to \infty. \end{array}\right\} \tag{14}$$

Now modify the coefficient matrix (10) by replacing $1+\eta_0$ by ρ_0, η_r by ρ_r and $\mathcal{J}_{0s}$ by P_{0s}. This gives a coefficient matrix P_{rs} of positive terms, such that each term is greater than the modulus of the corresponding term in (10) and each row forms a decreasing sequence of positive terms.

Now P_{n0} is formed from the elements of the second column according to the rule,

$$nP_{n0} = \rho_0 P_{n-1,1} + \rho_1 P_{n-2,2} + \ldots + \rho_{n-1}P_{01}. \tag{15}$$

But $\Sigma\rho_r$ converges, and hence there is a certain N_0 such that

$$\sum_0^{n-1} \rho_r < n \quad \text{if} \quad n > N_0.$$

Hence

$$\begin{aligned} P_{n0} &< \max(P_{01}, P_{11}, \ldots P_{n-1,1}) \\ &< \max(P_{00}, P_{10}, \ldots P_{n-1,0}), \end{aligned}$$

i.e. the sequence P_{n0} is bounded for all n. It immediately follows that the whole

coefficient matrix is bounded by the upper bound of the sequence P_{n0}. Let this number be a. Then the coefficient matrix is less than

$$\begin{matrix} a & a & a & a & \ldots \\ a & a & a & a & \ldots \\ a & a & a & a & \ldots \\ . & . & . & . & . \end{matrix}$$

Writing in powers of $\cos\theta$ and τ, we find that the double series so formed is summable by rows or columns to

$$a\frac{1}{1-\cos\theta}\,\frac{1}{1-\tau},$$

provided that

$$-1 < \cos\theta < 1 \qquad \text{and} \qquad 0 \leqslant \tau < 1.$$

It therefore follows that the double series whose coefficients are given by (10) is absolutely convergent in this region at least.

It is now necessary to show that there is no inconsistency in the assumption that $\Sigma\,|\,\eta_r\,|$ converges.

Lemma.—If $b_0, b_1, \ldots b_s \ldots$ is a bounded sequence of positive terms, and if $\rho_0, \rho_1, \ldots$ is a decreasing sequence of positive terms such that $\Sigma\rho_r$ converges, then

$$\frac{\rho_0 b_s + \rho_1 b_{s-1} + \ldots + \rho_s b_0}{s} \to 0 \quad \text{as} \quad s \to \infty.$$

Let p be a positive integer having the following properties:—

$$p \to \infty \quad \text{as} \quad s \to \infty, \qquad p/s \to 0 \quad \text{as} \quad s \to \infty.$$

Then

$$\frac{1}{s}\left|\sum_0^s \rho_r b_{s-r}\right| < \frac{1}{s}\left|\sum_0^p \rho_r b_{s-r}\right| + \frac{1}{s}\left|\sum_p^s \rho_r b_{s-r}\right|.$$

Let A be an upper bound of the sequence b_s. Then

$$\frac{1}{s}\left|\sum_0^p \rho_r b_{s-r}\right| < \frac{1}{s}\rho_0 p\,.A, \qquad \frac{1}{s}\left|\sum_p^s \rho_r b_{s-r}\right| < \frac{1}{s}A\sum_p^s \rho_r,$$

and each part can be made as small as we please by taking s large enough.

By equation (15) and this lemma we see that

$$P_{rs} \to 0 \quad \text{as} \quad r \to \infty,$$

and hence

$$\mathcal{J}_{rs} \to 0 \quad \text{as} \quad r \to \infty.$$

Now write

$$M_{rs} = \frac{\mathcal{J}_{r,\,s+2}}{3} + \frac{\mathcal{J}_{r,\,s+4}}{5} + \ldots \tag{16}$$

Then M_{rs} satisfies the same recurrence relations as $\mathcal{J}_{rs}$, and

$$M_{0s} = \frac{\mathcal{J}_{0,\,s+2}}{3} + \frac{\mathcal{J}_{0,\,s+4}}{5} + \ldots$$

converges and tends to zero as $s \to \infty$.*

* *Note.*—$\Sigma\mathcal{J}_{0s}$ converges and hence so does $\mathcal{J}_{00} + \mathcal{J}_{01} + \frac{1}{3}(\mathcal{J}_{02} + \mathcal{J}_{03}) + \frac{1}{5}(\mathcal{J}_{04} + \mathcal{J}_{05}) + \ldots$ $\frac{\mathcal{J}_{02}}{3} + \frac{\mathcal{J}_{04}}{5} + \ldots$ converges because $\mathcal{J}_{\nu}'$ exists at the surface. Hence by subtraction these properties of M_{0s} can be deduced.

We can prove the same theorems about M_{rs} as we did about $\mathcal{J}_{rs}$, and hence by (11) $\mathcal{J}_{\nu}'$ converges for $0 \leqslant \tau < 1$.

Now consider equation (12) and, for clarity only, write ϵ and δ zero. This equation becomes

$$0 = \mathcal{J}_{r0} - \sum_{s=0}^{r} \eta_{r-s} M_{s0} - a_r. \tag{17}$$

Now $\mathcal{J}_{r0}$ and a_r tend to zero as r tends to infinity, so that we must have

$$\left| \sum_{s=0}^{r} \eta_{r-s} M_{s0} \right| \to 0 \quad \text{as} \quad r \to \infty.$$

Form a matrix of quantities N_{rs} from the M_{rs} in the same way as we formed P_{rs} from the $\mathcal{J}_{rs}$. Then

$$\left| \sum_{s=0}^{r} \eta_{r-s} M_{s0} \right| < \left| \sum_{s=0}^{r} \rho_{r-s} N_{s0} \right|.$$

Then $N_{s0} \to 0$ as $s \to \infty$, so that if s is large enough $N_{s0} < \epsilon$. Take a positive integer p such that

$$p \to \infty \quad \text{as} \quad r \to \infty, \qquad p/r \to 0 \quad \text{as} \quad r \to \infty.$$

Then

$$\left| \sum_{s=0}^{r} \rho_{r-s} N_{s0} \right| < \left| \sum_{s=0}^{p} \rho_{r-s} N_{s0} \right| + \left| \sum_{p}^{r} \rho_{r-s} N_{s0} \right|.$$

Let B be an upper bound of N_{s0}. Then

$$\left| \sum_{s=0}^{p} \rho_{r-s} N_{s0} \right| < B \sum_{s=0}^{p} \rho_{r-s},$$

which $\to 0$ as $r \to \infty$ because $\Sigma \rho_r$ converges, and

$$\left| \sum_{p}^{r} \rho_{r-s} N_{s0} \right| < \epsilon \sum_{0}^{p} \rho_r$$

if r is large enough. Hence all the terms in (17) tend to zero, and further the coefficient of η_r, viz. M_{00}, is finite.

Summary

The paper shows that if ϵ and δ are known constants, then if one solution for η has been found from a given set of observations of centre to limb variations of a line contour, there is no second solution, provided that the solution found is valid at least to an optical depth greater than unity.

The explicit enunciation of this problem and a practical solution in the case of the *Mg b* lines in the solar spectrum has been given by Professor H. H. Plaskett *, and the author wishes gratefully to acknowledge the stimulus to consider the theoretical aspect of the problem which came from his work. Acknowledgment must also be made for many illuminating discussions with Mr. G. Leslie Camm.

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ON THE SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

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(Communicated by Professor H. H. Plaskett, F.R.S.)

1. *Introduction.*—The equation of radiative transfer

$$I_\lambda(0, \theta) = \sec\theta \int_0^\infty B_\lambda(\tau) e^{-\tau \sec\theta} d\tau \qquad (1.1)$$

has been solved numerically by methods depending upon trial and error *, but hitherto no mathematical expression for $B_\lambda(\tau)$ has been found, which will reproduce values of $I_\lambda(0, \theta)$ differing from the observed values by less than the experimental error. Now if

$$p = \sec\theta, \qquad \phi(p) = I_\lambda(0, \theta),$$

equation (1.1) takes the form

$$\phi(p) = p \int_0^\infty B_\lambda(\tau) e^{-p\tau} d\tau. \qquad (1.2)$$

This is Laplace's integral equation, the solution of which is known when $\phi(p)$ is known and certain conditions are satisfied. Thus the equation (1.1) can be solved when the functional form for $I_\lambda(0, \theta)$ has been determined, provided that the appropriate conditions are satisfied. The object of the investigation, described in this paper, was therefore the discovery of a functional form for $I_\lambda(0, \theta)$ and thence the solution of the equation of transfer. The accuracy of the solution depends upon the correctness of the expression for $I_\lambda(0, \theta)$ over the whole interval $(0, \pi/2)$, but since verification of values for points near to the limb of the Sun is not at present possible †, no estimate of the accuracy of the solution can be made.

2. *The Functional Form of* $I_\lambda(0, \theta)$.—The values of

$$I_\lambda(\theta) = I_\lambda(0, \theta)/I_\lambda(0, 0)$$

found by Moll, Burger and van der Bilt ‡ for λ 5500 were first considered, and after several unsuccessful trials, it was found that when $\{I_\lambda(\theta)\}^{-2}$ was plotted against sec θ, the points lay on a straight line of slope ·92. Since $I_\lambda(0) = 1$, this gave

$$I_\lambda(\theta) = (\cdot 92 \sec\theta + \cdot 08)^{-1/2} \qquad (\lambda = 5500). \qquad (2.1)$$

Starting with this formula, the values of $I_\lambda(\theta)$ for several values of θ were calculated and compared with the observed values. These are given in the second and third columns respectively of Table I. Column 4 gives the difference, which is in every case less than 1 per cent.—the estimated accuracy of the observations. It is, in fact, less than 0·4 per cent.

On plotting $\{I_\lambda(\theta)\}^{-2}$ against sec θ for other wave-lengths it was found that the points lay on curves, but a straight line was obtained in each case when $\{I_\lambda(\theta)\}^{-1/\alpha}$ was plotted against sec θ, α being positive. Denoting the slope of the corresponding line by m, this gave

$$I_\lambda(\theta) = \{m \sec\theta + 1 - m\}^{-\alpha} \qquad (\alpha > 0). \qquad (2.2)$$

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* H. H. Plaskett, *M.N.*, **96**, 402, 1936.

† See the end of section 2 for a further discussion of this point.

‡ *B.A.N.*, **3**, 83, 1925.

The values of $I_\lambda(\theta)$ calculated from this formula and the corresponding observed values are given in Tables II to VI.*

TABLE I

$\lambda = 5500$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·5	0·828	0·826	+·002
2·0	·722	·719	+·003
2·5	·648	·647	+·001
3·0	·593	·594	−·001
3·5	·550	·551	−·001
4·0	·516	·517	−·001

TABLE II

$\lambda = 4500, \quad \alpha = \cdot 8, \quad m = \cdot 694$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·40	0·822	0·822	·000
2·50	·565	·561	+·004
2·86	·514	·514	·000
3·33	·463	·471	−·008
5·00	·345	·337	+·008

TABLE III

$\lambda = 5000, \quad \alpha = \cdot 64, \quad m = \cdot 78$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·40	0·840	0·838	+·002
2·00	·691	·688	+·003
2·50	·609	·616	−·007
3·33	·515	·520	−·005
5·00	·405	·400	+·005

TABLE IV

$\lambda = 5060, \quad \alpha = \cdot 61, \quad m = \cdot 82$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·556	0·795	0·788	+·007
1·743	·748	·744	+·004
2·000	·694	·696	−·002
2·366	·632	·637	−·005
2·924	·561	·562	−·001
3·864	·478	·481	−·003

TABLE V

$\lambda = 5960, \quad \alpha = \cdot 425, \quad m = \cdot 999$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·556	0·829	0·825	+·004
1·743	·790	·788	+·002
2·000	·745	·745	·000
2·366	·693	·694	−·001
2·924	·634	·634	·000
3·864	·563	·560	+·003

TABLE VI

$\lambda = 6700, \quad \alpha = \cdot 39, \quad m = \cdot 967$

sec θ	$I_\lambda(\theta)$ Cal.	$I_\lambda(\theta)$ Obs.	Diff.
1·556	0·846	0·840	+·006
1·743	·810	·809	+·001
2·000	·768	·772	−·004
2·366	·720	·720	·000
2·924	·664	·669	−·005
3·864	·596	·597	−·001

* In Tables IV, V and VI the observed values are taken from tables given by Raudenbusch in *A.N.*, 263, 300, 1938.

Fig. 1 shows the variation of a with λ. There is clearly some functional relation between them, but so far no mathematical expression for it has been discovered. It has been found, however, that the equation

$$\lambda am = 2520 \tag{2.3}$$

is at least approximately true. Values are given in Table VII.

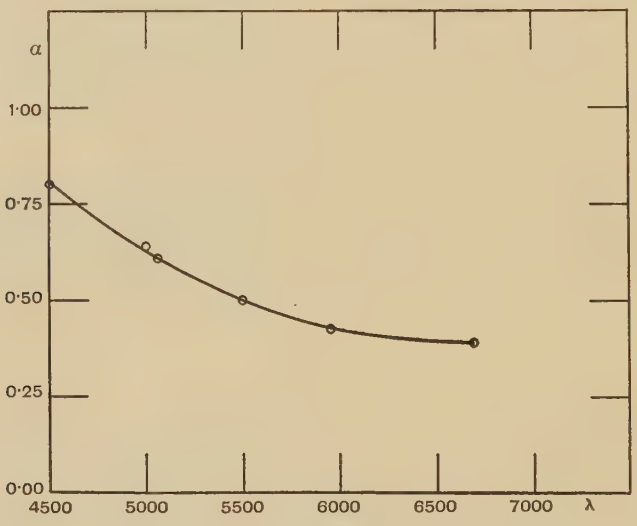

Fig. 1.—*Variation of a with wave-length.*

Table VII

λ	a	m	λam
4500	0·800	0·649	2498
5000	·640	·780	2496
5060	·610	·820	2531
5500	·500	·920	2530
5960	·425	·999	2530
6700	·390	·967	2527

An empirical formula to fit the Moll, Burger, van der Bilt curves has been given by Hertzsprung.* When λ is measured in angstroms, his formula is

$$I_\lambda(\theta) = \cos^{\frac{2700}{\lambda}} \theta. \tag{2.4}$$

If we take $m = 1$ in (2.2) and (2.3), these give

$$I_\lambda(\theta) = \cos^{\frac{2520}{\lambda}} \theta.$$

This suggests that (2.4) is a first approximation to (2.2), which agrees much more closely with the observed values.

The values of $I_\lambda(\theta)$ compared in Tables I to VI are all for values of θ less than 78° 30′. Accurate observations for larger values of θ can only be obtained at the time of an eclipse.

* *B.A.N.*, **8**, 140, 1937.

Minnaert * has made such observations, but he says that his only reliable results were those obtained for total radiation during the eclipse of 1912. These he represents by the formula

$$I_\lambda(\theta) = 0{\cdot}39 + 10\frac{\Delta r}{r} - \frac{0{\cdot}39}{\left(1 + 2200\frac{\Delta r}{r}\right)^2}, \tag{2.5}$$

where $\frac{\Delta r}{r} = 1 - \sin\theta$. Now when $\sin\theta = {\cdot}98$, equation (2.1) gives $I_\lambda(\theta) = {\cdot}46$. If we change ·39 to ·26 to make (2.5) agree with (2.1) when $\sin\theta = {\cdot}98$, and then calculate the values of $I_\lambda(\theta)$ from the two equations, we get the following results:—

$\sin\theta$	·98	·99	·995	·999	·9999
Eqn. (2.5)	·46	·36	·31	·24	·09
Eqn. (2.1)	·46	·39	·33	·22	·12

This suggests that (2.1), and similarly (2.2), represents $I_\lambda(\theta)$ right up to the edge of the Sun.

3. *The Solution of Equation* (1.1).—Various mathematical theorems exist, which could be quoted for the solution of equation (1.2). In Theorem 1 some knowledge of $B_\lambda(\tau)$ is assumed, but the conditions chosen are ones which the temperature distribution might be expected to possess. They are satisfied if the distribution is either isothermal or linear. If $B_\lambda(\tau)$ satisfies these conditions there is a uniqueness theorem (Theorem 2), which implies that, corresponding to one law of darkening, there is only one possible temperature distribution.

Theorem 1.—Let $B_\lambda(\tau)$ be a positive, continuous, non-decreasing function of τ for $\tau > 0$ and let $e^{-c\tau} B_\lambda(\tau)$, where $0 < c < 1$, be ultimately non-increasing. Then

$$\phi(z) = z\int_0^\infty e^{-zt} B_\lambda(t)\,dt$$

is an analytic function of the complex variable z, which is regular for $R(z) > c$, and if $c' > c$,

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{c'-iN}^{c'+iN} \frac{\phi(z)}{z} e^{z\tau}\,dz = \begin{cases} B_\lambda(\tau) & (\tau > 0), \\ \frac{1}{2}\lim_{\tau\to+0} B_\lambda(\tau) & (\tau = 0), \\ 0 & (\tau < 0). \end{cases} \tag{3.1}$$

Theorem 2.—Let $B_\lambda(\tau)$ satisfy the conditions of Theorem 1 and let

$$\int_0^\infty e^{-pt} B_\lambda(t)\,dt = \phi(p)/p,$$

where $\phi(p)$ is defined for all real p such that $1 < p < \infty$. Then $B_\lambda(\tau)$ is given uniquely by equation (3.1), except perhaps when $\tau = 0$.

In these theorems $B_\lambda(\tau)$ is really considered over the interval $(-\infty, \infty)$, the value for $\tau < 0$ being zero. If $B_\lambda(0) = 0$, the function is continuous, but if $B_\lambda(0) \neq 0$, there is a discontinuity when $\tau = 0$. Consider, for example, an isothermal star for which $B_\lambda(\tau) = B$. Then $\phi(z) = B$, and the value of the integral on the left-hand side of (3.1) is $B(\tau > 0)$ and $0(\tau < 0)$, but when $\tau = 0$ it is $\frac{1}{2}B$. This is half the value to which $B_\lambda(\tau)$ tends as $\tau \to 0$ through positive values. Since it is this latter value which matters in the physical problem, we can ignore the value of the integral when $\tau = 0$, considering the integral only for $\tau > 0$ and its limit as $\tau \to 0$.

Using the notation of section 1, we have from (2.2)

$$\phi(p) = \kappa(p + a)^{-\alpha}, \tag{3.2}$$

* *M.N.*, **89**, 197, 1928.

where

$$a=\frac{1-m}{m}, \qquad \kappa=m^{-a}I_\lambda(0,0). \tag{3·3}$$

We therefore take $\phi(z)=\kappa(z+a)^{-a}$ in (3.1). This gives

$$B_\lambda(t)=\frac{\kappa}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{zt}}{z(z+a)^a}dz, \qquad (t>0).$$

Differentiating with respect to t, we have

$$B_\lambda'(t)=\frac{\kappa}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{zt}}{(z+a)^a}dz, \qquad (t>0),$$

and putting $z+a=\zeta$, this gives

$$\begin{aligned}B_\lambda'(t)&=\frac{\kappa e^{-at}}{2\pi i}\int_{a+c'-i\infty}^{a+c'+i\infty}\frac{e^{\zeta t}}{\zeta^a}d\zeta\\&=\frac{\kappa e^{-at}t^{a-1}}{\Gamma(a)}\end{aligned} \tag{3·4}$$

by a well-known result, since $a>0$. Hence

$$B_\lambda(\tau)=\frac{\kappa}{\Gamma(a)}\int_0^\tau e^{-at}t^{a-1}dt. \tag{3·5}$$

Substituting the series for e^{-at} and integrating term by term, we have

$$B_\lambda(\tau)=\frac{\kappa}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}\frac{\tau^{n+a}}{n+a};$$

i.e. by (3.3),

$$B_\lambda(\tau)=\frac{I_\lambda(0,0)}{\Gamma(a)}\left(\frac{\tau}{m}\right)^a\sum_{n=0}^{\infty}\frac{(-a\tau)^n}{n!\,(n+a)} \qquad \left(a=\frac{1-m}{m}\right). \tag{3·6}$$

The mathematical operations used in the evaluation of this integral are all easily justified, and it can be shown that the function defined by (3.6) satisfies all the conditions imposed on $B_\lambda(\tau)$ in Theorem 1. Thus it follows from Theorem 2 that if (2.2) gives the true law of darkening, then (3.6) gives the correct temperature distribution. The series converges for every value of τ and, since a is always small, only a few terms are required for computation.

As $\tau\to 0$, $B_\lambda(\tau)\to 0$ and if $a>0$ (as it is in each of the cases considered) it follows from (3.5) that $B_\lambda(\tau)\to\kappa a^{-a}=I_\lambda(0,0)/(1-m)^a$ as $\tau\to\infty$. These values are not acceptable to astrophysicists and one is therefore forced to the conclusion that either equation (2.2) is wrong or that some factor, which seriously affects the values of $B_\lambda(\tau)$ at 0 and ∞, has been ignored. Now the values of θ in (1.1) which are important when τ is very small or very large, are those which make sec θ large—*i.e.* the values near to the limb of the Sun, where the curvature of the Sun is no longer negligible. Since equation (1.1) is for a plane distribution, it follows that the values of $B_\lambda(\tau)$ derived from it are not reliable for very small and very large values of τ. Since (from equation (3.4)) $B_\lambda'(\tau)\to\infty$ as $\tau\to 0$, $B_\lambda(\tau)$ increases initially with infinite rapidity and we therefore get values of $B_\lambda(\tau)$ which are by no means negligible when τ is quite small (*e.g.* $\tau=0{\cdot}1$). These values should be fairly reliable.

4. *The Values of $B_\lambda(\tau)$ for λ 5500.*—In this case a numerical solution to equation (1.1) has been obtained by Plaskett *, and it is therefore of the utmost importance to compare his solution with that obtained above.

* *Loc. cit.*, p. 419.

When $\alpha = 1/2$, equation (3.5) becomes, after putting $at = u^2$,

$$B_\lambda(\tau) = \frac{I_\lambda(0, 0)}{(1-m)^{1/2}} \cdot \frac{2}{\pi^{1/2}} \int_0^{\sqrt{a\tau}} e^{-u^2} du, \qquad (4.1)$$

where $I_\lambda(0, 0) = \cdot 5923, \quad m = \cdot 92, \quad a = \cdot 087.$

In this form the values of $B_\lambda(\tau)$ can be calculated from mathematical tables. Some of these are given in the second column of Table VIII.

Plaskett divided the solar atmosphere according to τ into 17 layers and obtained an approximate value of $B_\lambda(\tau)$ for each layer. The limits for these layers are given in the third column of Table VIII and the corresponding value for $B_\lambda(\bar{\tau})$ in the fourth column. $\bar{\tau}$ denotes the mean value of τ for the layer.

TABLE VIII

Temperature Distribution

τ	$B_\lambda(\tau)$	$\tau_1 - \tau_2$	$B_\lambda(\bar{\tau})$	τ	$B_\lambda(\tau)$	$\tau_1 - \tau_2$	$B_\lambda(\bar{\tau})$
0·1	0·219	0·0–0·2	0·206	1·9	·904	1·8– 2·0	·851
0·3	·379	0·2–0·4	·280	2·2	·971	2·0– 2·4	·934
0·5	·487	0·4–0·6	·561	2·6	1·05	2·4– 2·8	1·038
0·7	·572	0·6–0·8	·671	3·0	1·11	2·8– 3·2	1·148
0·9	·645	0·8–1·0	·671	3·6	1·20	3·2– 4·0	1·289
1·1	·708	1·0–1·2	·671	4·4	1·30	4·0– 4·8	1·473
1·3	·765	1·2–1·4	·671	6·4	1·48	4·8– 8·0	1·877
1·5	·817	1·4–1·6	·734	10·0	1·70	8·0–12·0	2·478
1·7	·867	1·6–1·8	·792	∞	2·09		

In fig. 2 the curve is the graph of the function (4.1), and the isolated points show the values found by Plaskett. Clearly the solution (4.1) fails to reproduce the isothermal

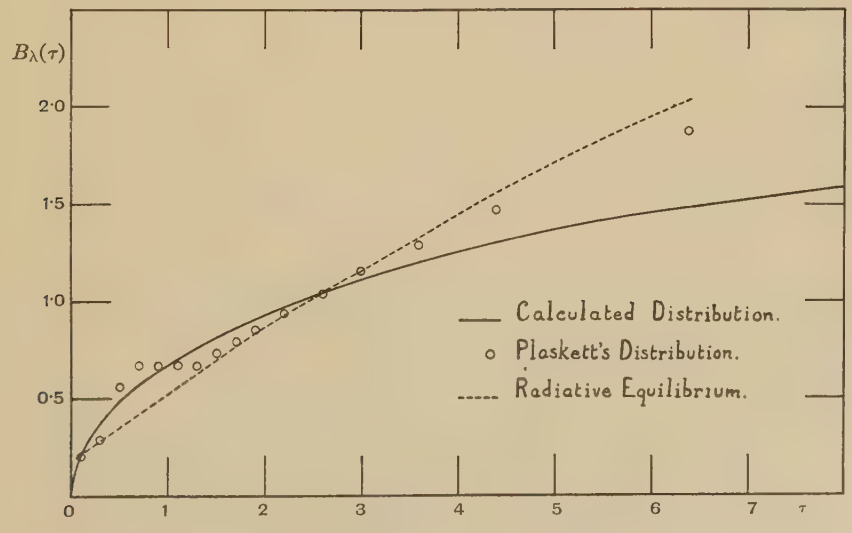

FIG. 2.—*Temperature Distribution.*

layer discovered by Plaskett. The dotted line in the figure shows the distribution which would hold in the event of radiative equilibrium. This is included for comparison.

The differences between the two solutions probably arise in two ways. The main differences occur for large values of τ and in the region of the isothermal layer. Those for large values of τ are undoubtedly due to the fact that Plaskett considered only the known values of $I_\lambda(\theta)$ for $1 \leqslant \sec\theta \leqslant 4$, whereas the mathematical solution depends on the correctness of the expression (2.1) over the whole interval $(0, \pi/2)$. The differences in the region of the isothermal layer may arise from the same cause, or they may be partly or wholly due to the method by which Plaskett arrived at his results, a discussion of which follows.

After making the substitution

$$y = e^{-\tau \sec\theta}, \tag{4.2}$$

equation (1.1) becomes

$$I_\lambda(0, \theta) = \int_0^1 B_\lambda(\tau) \, . \, dy. \tag{4.3}$$

Plaskett replaced this integral by a summation

$$I_\lambda(0, \theta) = \sum_{i=1}^{n} B_\lambda(\bar{\tau}_i) \, . \, \Delta y_i, \tag{4.4}$$

where $\bar{\tau}_i$ was the mean value of τ over the interval $(y_i, y_i + \Delta y_i)$. Values of $B_\lambda(\bar{\tau})$ were found by trial which, when substituted into (4.4), gave the known value of $I_\lambda(0, \theta)$ for the particular value of θ in question.

In fig. 3 the values of $B_\lambda(\tau)$ calculated from (4.1) have been plotted against y; thus it follows from (4.3) that the area under the curve denotes $I_\lambda(0, \theta)$. (In the case shown,

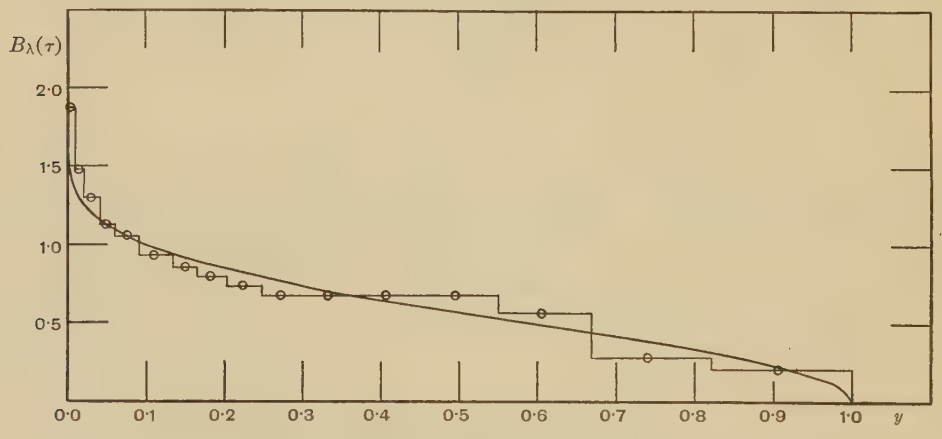

FIG. 3.—*Temperature Distribution as a Function of* $y = e^{-\tau \sec\theta}$.

θ has been taken to be zero as this is the most important case and the graphs obtained for other values of θ are similar.) Plaskett's values for $B_\lambda(\bar{\tau})$ have also been plotted, and it follows from (4.4) that the area under the broken line is also $I_\lambda(0, \theta)$. It is obvious that these areas are at least approximately equal.

The curve in fig. 3 has a point of inflexion when $y = \cdot 58$, and for $\cdot 2 < y < \cdot 9$ it approximates to a straight line. If, therefore, a horizontal line, with its mid-point on the curve, were drawn anywhere between $y = \cdot 2$ and $y = \cdot 9$, and values for $B_\lambda(\bar{\tau})$ for this interval chosen to lie along it, a good approximation for $I_\lambda(0, \theta)$ would result. It seems probable, therefore, that the isothermal layer, which here stretches from $y = \cdot 247$ to $y = \cdot 549$, could occupy a different position equally well. Thus, although there are physical reasons for a belief in the existence of an isothermal layer, it appears to arise mathematically from the method of approximation used in the solution of the equation (1.1).

Summary

1. The equation

$$\frac{I_\lambda(0, \theta)}{I_\lambda(0, 0)} = (m \sec \theta + 1 - m)^{-\alpha}, \qquad (\alpha > 0)$$

is found to be true, within the limits of experimental error, for six different wave-lengths, α, m and λ being related by the equation

$$\lambda \alpha m = 2520$$

when λ is measured in angstroms.

2. The equation of radiative transfer is then solved mathematically, the solution being

$$B_\lambda(\tau) = \frac{I_\lambda(0, 0)}{\Gamma(\alpha)} \left(\frac{\tau}{m}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-a\tau)^n}{n!\,(n+\alpha)}, \qquad \left(a = \frac{1-m}{m}\right).$$

3. The solution for λ 5500 is discussed in detail and the results compared with those already obtained by Plaskett.

In conclusion I should like to express my thanks to Professor H. H. Plaskett and to Professor E. A. Milne, from both of whom I have received several long and helpful letters on the subject. To Professor Plaskett I am indebted for the discussion on the darkening near to the limb of the Sun, which is given at the end of section 2. I should also like to thank members of the University Observatory for their interest in the problem and for helpful criticisms.

St. Hugh's College, Oxford:
1940 *August* 28.

ON THE FIGURES OF THE EARTH AND MOON

(Second Paper)

Harold Jeffreys

In a former paper * I used data for the lunar perturbations and librations to determine the ellipticities of the Earth and Moon. Almost immediately afterwards E. W. Brown published a correction to the solar contributions to the motions of the Moon's node and perigee †, which entailed some corrections to my determinations; I gave a revised solution ‡, and another was given, with somewhat different observational data, by H. Spencer Jones. § Recently I have made a new discussion of the observations of gravity over the Earth's surface, in an attempt to determine the harmonics of low order other than the main ellipticity term. ‖ The observations turned out to be unsuitably distributed for such a determination, but did yield a new estimate of the main ellipticity term in gravity, with an uncertainty that appears to be valid. It rests on a hypothesis of randomness that will need further check, but it is less extreme than has usually been made in such determinations and is consistent with the data so far as they exist. It is now worth while to make a complete new solution for the constants involved. The notation is as in my previous paper.

Previously I had, for the parts of the annual motions of the perigee and node to be attributed to the figures of the Earth and Moon,

$$d\varpi = +6''\cdot516 \pm 0\cdot052; \qquad d\Omega = -6''\cdot187 \pm 0\cdot031 \quad \text{(probable errors)}.$$

Brown adds $+0''\cdot14$ and $-0''\cdot04$ respectively to the calculated solar effects; while Spencer Jones adds $+0''\cdot035$ and $-0''\cdot032$ to the observed values. Hence my equations (1), (2), (3) must be replaced by

$$+3891\cdot1\mathcal{J} + 390\mathcal{J}' - 1027K' = +6\cdot411 \pm 0\cdot039, \tag{1}$$

$$-3647\cdot5\mathcal{J} - 470\mathcal{J}' - 235K' = -6\cdot179 \pm 0\cdot039. \tag{2}$$

The probable errors stated are obtained by combining the separate contributions stated by Spencer Jones. The greater parts represent the uncertainty of the calculation. My equation (5) needs no change; it is

$$1\cdot0047\mathcal{J}' + 0\cdot4976K' = 0\cdot0003746 \pm 0\cdot0000011, \tag{3}$$

and the monthly term in the Moon's latitude gives

$$\mathcal{J} = 0\cdot0016328 \pm 0\cdot0000102. \tag{4}$$

The conflicting evidence about the ratio f for the Moon appears to have been resolved by Belkovitch ¶, who, from a new discussion of Michailovski's observations, gets $0\cdot84 \pm 0\cdot08$. Hayn's value, as revised by de Sitter, was $0\cdot85 \pm 0\cdot07$, while Franz and Stratton, from Schlüter's observations, got $0\cdot50 \pm 0\cdot03$. Thus Belkovitch confirms Hayn. To avoid overweighting of selected data, however, it seems undesirable to

* *M.N.*, **97**, 3–15, and *Geophys. Suppl.*, **4**, 1–13, 1937.
† *M.N.*, **97**, 126, 1937.
‡ *Nature*, **139**, 471, 1937.
§ *M.N.*, **97**, 406–9, 1937.
‖ *M.N., Geophys. Suppl.* (in the Press).
¶ *Engelhardt Obs. Bull.*, Kazan, No. 10, 28, 1936.

combine their results, and I shall use Belkovitch's solution alone. We have

$$f=\frac{\mathcal{J}'-\frac{1}{2}K'}{\mathcal{J}'+\frac{1}{2}K'}=0{\cdot}84\pm0{\cdot}08, \tag{5}$$

whence

$$K'/\mathcal{J}'=0{\cdot}174\pm0{\cdot}076. \tag{6}$$

My determination of the coefficient of $\sin^2\phi$ in gravity is $0{\cdot}0052866\pm0{\cdot}0000055$ (p.e.). De Sitter's 0·0052884 corresponds to $\mathcal{J}=0{\cdot}0016388$ and $e=\cdot0033675$; hence my value gives

$$\mathcal{J}=0{\cdot}0016406\pm0{\cdot}0000055. \tag{7}$$

To solve it is convenient to write

$$\mathcal{J}=0{\cdot}0016400+10^{-4}j\,;$$
$$\mathcal{J}'+\tfrac{1}{2}K'=0{\cdot}00037+10^{-4}\lambda;$$
$$K'=10^{-4}k.$$

The equations become

				O − C	
$0{\cdot}389j$	$+0{\cdot}039\lambda$	$-0{\cdot}122k$	$=-0{\cdot}114\pm0{\cdot}039$	−0·017	(1′)
$-0{\cdot}365j$	$-0{\cdot}047\lambda$		$=+0{\cdot}009\pm0{\cdot}039$	+0·003	(2′)
	λ	$-0{\cdot}005k$	$=+0{\cdot}029\pm0{\cdot}011$	0·000	(3′)
j			$=-0{\cdot}072\pm0{\cdot}102$	−0·049	(4′)
	$0{\cdot}16\lambda$	$-k$	$=-0{\cdot}59\ \pm0{\cdot}30$	+0·14	(5′)
j			$=+0{\cdot}006\pm0{\cdot}055$	+0·029	(7′).

In (5′) the uncertainty of λ is neglected.

The solution is

$$j=-0{\cdot}023\pm0{\cdot}042, \tag{8}$$

$$\lambda=+0{\cdot}033\pm0{\cdot}011, \tag{9}$$

$$k=+0{\cdot}73\ \pm0{\cdot}23, \tag{10}$$

and the residuals give

$$\chi^2=0{\cdot}92\times(0{\cdot}675)^2=0{\cdot}42$$

on 3 degrees of freedom. This is very small, $P=0{\cdot}9$, so that the data are more consistent than would be expected from their intrinsic accuracy. The factor $(0{\cdot}675)^2$ is needed because comparison has been made with probable and not standard errors.

Then

$$\mathcal{J}=0{\cdot}0016377\pm0{\cdot}0000042\ (0{\cdot}0000062\ \text{s.e.}), \tag{11}$$

and hence the coefficient of $g_0\sin^2\phi$ in gravity is $0{\cdot}0052895\pm0{\cdot}0000042$ and the ellipticity is

$$\epsilon=0{\cdot}0033664\pm0{\cdot}0000042\ (0{\cdot}000062\ \text{s.e.}), \tag{12}$$

$$1/\epsilon=297{\cdot}05\pm0{\cdot}38\ (0{\cdot}56\ \text{s.e.}). \tag{13}$$

The last result may be compared with de Sitter's 296·75 ± 0·10 (p.e.) derived from the precessional constant on the hypothesis of hydrostatic stress in the Earth, and with Heiskanen's $297{\cdot}8\pm0{\cdot}4$ (s.e.) with allowance for the longitude terms and $298{\cdot}3\pm0{\cdot}3$ without it, both based on the hypothesis that departures from the formula in 1° squares are uncorrelated. The present value has about the same apparent accuracy as Heiskanen's, and there is good reason to suppose that this accuracy is genuine. The International gravity formula (Potsdam standard) will be modified to

$$g=978{\cdot}049(1+0{\cdot}0052895\sin^2\phi-0{\cdot}0000059\sin^2 2\phi)$$

to keep the mean value consistent with my solution from the gravity observations. The mean value itself needs a correction since it is now known that the absolute measurement

of gravity at Potsdam was about 17 ± 2 mgal. too high.* The first factor should then be reduced to 978·032 gals.; but, as remarked by Browne and Bullard, it is undesirable for most geophysical purposes to make any departure from the present international formula as a comparison standard, at any rate until the connections between Potsdam, Cambridge, and the National Physical Laboratory have been strengthened.

The constants for the Moon are now

$$\mathcal{J}' + \tfrac{1}{2}K' = 0{\cdot}0003733 \pm 0{\cdot}0000011 \ (0{\cdot}0000016 \text{ s.e.}),$$
$$K' = 0{\cdot}000073 \pm 0{\cdot}000023 \ (0{\cdot}000034 \text{ s.e.}),$$
$$f = 0{\cdot}804 \pm 0{\cdot}062 \ (0{\cdot}092 \text{ s.e.}).$$

It is evidently desirable to use $\mathcal{J}' + \frac{1}{2}K'$ explicitly rather than $\mathcal{J}'$ on account of its small uncertainty; its definition is

$$\mathcal{J}' + \tfrac{1}{2}K' = \frac{3(C' - B')}{2M'b'^2}.$$

K' also is more directly related to the observations than f.

An outstanding difficulty is the disagreement between the Moon's dynamical and visual parallaxes. According to Lambert † the former is the larger by 0″·16 if the ellipticity is taken as 1/297, and agreement is obtained if it is taken as 1/293·5. The dynamical parallax is decreased by $1''{\cdot}17\Delta g$ if g is in gals., and therefore the change in the absolute measure of gravity will increase it by about 0″·020. The change therefore hardly affects the discrepancy, which, however, may well be due to an abnormal deflection of the vertical at Cape Town.

Summary

Six data relevant to the ellipticities of the Earth and Moon have been combined, namely, the secular motions of the node and perigee, the monthly term in the Moon's latitude, the inclination of the Moon's axis, the annual libration in longitude, and a new determination of the ellipticity term in gravity. The agreement is satisfactory.

* B. C. Browne and E. C. Bullard, *Proc. Roy. Soc.*, A, 110–7, 1940.
† *A.J.*, 38, 181–5, 1928.

THE SYSTEMATIC CORRECTIONS TO THE PROPER MOTIONS OF THE GENERAL CATALOGUE

W. M. Smart

1. The only comprehensive discussion, up to date, of the proper motions of the *General Catalogue* * was undertaken by R. E. Wilson and H. Raymond, who derived † the elements of the solar motion, the precessional corrections and the constants of galactic rotation as the result of two solutions, the first based on Schönfeld's equations ‡ in equatorial co-ordinates, and the second in galactic co-ordinates. Miss Emma Williams § has recently pointed out that the equatorial equations are incomplete so far as the rotational effect is concerned, for the double-wave term depending on Oort's constant A is omitted; in addition, a misinterpretation of Schönfeld's notation and the omission of a factor lead to certain adjustments, fortunately small, in the values of the constants to be determined. The equations in galactic co-ordinates are complete.

Wilson and Raymond finally arrived at values, based on their two solutions, of the quantities Δk, Δn (depending, in right ascension and declination, on the errors in Newcomb's precession and in the equinox), Oort's rotational constant, B, and the longitude, l_0, of the galactic centre; Oort's constant, A, was obtained from the solution in galactic co-ordinates alone, but the value finally adopted was that given by Plaskett and Pearce ‖ in their investigation on the O and B-type stars.

If we are concerned with the precessional and equinox corrections only, we require to apply to the proper-motion components, $\mu_\alpha \cos\delta$ and μ_δ, the quantities $-\Delta k \cos\delta - \Delta n \sin\alpha \sin\delta$ and $-\Delta n \cos\alpha$ respectively, so as to remove the systematic effects introduced by the slightly erroneous values of the fundamental constants. But to remove all the ascertainable systematic effects we require further to apply the rotational corrections; as Wilson and Raymond remark, the effects of these are now known almost, if not fully, as accurately as the precessional corrections. The numerical formulæ which they derive ¶ for the systematic corrections to the proper-motion components are, however, incomplete, and, although these are probably sufficiently accurate for investigations depending on the larger proper motions, they do not fully satisfy the requirements of investigations dealing with the small proper motions. In any event there is no reason why the best available systematic corrections should not be applied, and it is the object of this paper to provide the complete numerical data. We first consider the accurate formulæ required.

2. In galactic co-ordinates the components of proper motion due to galactic rotation are given by the well-known formulæ:

$$u \equiv \mu_l \cos b = A_1 \cos 2(l - l_0) \cos b + B_1 \cos b, \tag{1}$$

$$v \equiv \mu_b = -A_1 \sin 2(l - l_0) \sin b \cos b, \tag{2}$$

where (l, b) are the galactic co-ordinates of the star concerned and $A_1 = A/4{\cdot}74$, $B_1 = B/4{\cdot}74$, Oort's constants A and B being expressed as usual in km./sec./parsec.

* Carnegie Institution of Washington, 1937.
† *A.J.*, 47, 49, 1938.
‡ *V.J.S.*, 17, 1882; *A.J.*, 47, 52, 1938.
§ *A.J.*, 48, 84, 1939; see also note by Wilson and Raymond, *A.J.*, 48, 86, 1939.
‖ *Publ. D.A.O.* (*Victoria*), 5, Pt. 4, 1936.
¶ *A.J.*, 47, 57, 1937.

If we are to make the solutions in terms of equatorial co-ordinates we require to transform l and b in (1) and (2) into functions of α, δ, A_0 and D_0, where (A_0, D_0) are the equatorial co-ordinates of the galactic pole. One method is to make the transformation by way of the direction-cosines of the appropriate galactic rectangular axes with respect to the equatorial system; this was the procedure adopted by Schilt * and followed later by Holm.† In both investigations the numerical values of A_0 and D_0 (190° and +28° —the Harvard pole) were inserted at the outset in the direction-cosine formulæ. Literal formulæ do not seem to have been published before; they are readily derived as follows:—

Let P denote the pole of the equator, $W(A_0, D_0)$ the pole of the galactic equator and $X(\alpha, \delta)$ a star whose galactic co-ordinates are (l, b). Let ϕ denote the angle WXP which is analogous to position angle, being measured from 0° to 360° from the galactic meridian XW in the direction of increasing longitude.

Let U and V denote the equatorial components of the rotational effects on the proper motions. Then we have at once,

$$U = u\cos\phi - v\sin\phi, \tag{3}$$

$$V = u\sin\phi + v\cos\phi, \tag{4}$$

where u and v are given by (1) and (2). For convenience we write

$$\xi \equiv A_1\cos 2l_0, \qquad \eta \equiv A_1\sin 2l_0, \tag{5}$$

and we obtain

$$U = \xi(\cos 2l\cos b\cos\phi + \sin 2l\sin b\cos b\sin\phi) + \eta(\sin 2l\cos b\cos\phi - \cos 2l\sin b\cos b\sin\phi) + B_1\cos b\cos\phi, \tag{6}$$

$$V = \xi(\cos 2l\cos b\sin\phi - \sin 2l\sin b\cos b\cos\phi) + \eta(\sin 2l\cos b\sin\phi + \cos 2l\sin b\cos b\cos\phi) + B_1\cos b\sin\phi. \tag{7}$$

The transformation of (6) and (7) into equations involving equatorial co-ordinates only is effected by means of the following formulæ easily found from the triangle WPX:—

$$\cos l\cos b = \cos\delta\sin(\alpha - A_0), \tag{8}$$

$$\cos b\sin\phi = \cos D_0\sin(\alpha - A_0), \tag{9}$$

$$\sin b = \sin D_0\sin\delta + \cos D_0\cos\delta\cos(\alpha - A_0), \tag{10}$$

$$\sin l\cos b = \cos D_0\sin\delta - \sin D_0\cos\delta\cos(\alpha - A_0), \tag{11}$$

$$\cos b\cos\phi = \sin D_0\cos\delta - \cos D_0\sin\delta\cos(\alpha - A_0), \tag{12}$$

$$\sin l\cos\phi - \cos l\sin b\sin\phi = -\cos(\alpha - A_0), \tag{13}$$

$$\cos l\cos\phi + \sin l\sin b\sin\phi = \sin D_0\sin(\alpha - A_0), \tag{14}$$

$$\sin l\sin\phi + \cos l\sin b\cos\phi = \sin\delta\sin(\alpha - A_0). \tag{15}$$

For example, if a denotes the coefficient of ξ in (6), we have, using (8), (14) and (12),

$$\begin{aligned} a &= 2\cos l\cos b(\cos l\cos\phi + \sin l\sin b\sin\phi) - \cos b\cos\phi \\ &= 2\cos\delta\sin(\alpha - A_0)\{\sin D_0\sin(\alpha - A_0)\} - \sin D_0\cos\delta + \cos D_0\sin\delta\cos(\alpha - A_0) \\ &= \cos D_0\sin\delta\cos(\alpha - A_0) - \sin D_0\cos\delta\cos 2(\alpha - A_0). \end{aligned}$$

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* *Proc. N.A.S.*, 13, 642, 1927.
† *Lund Medd.*, Ser. II, No. 55, 44, 1930.

Proceeding in this way we easily arrive at the formulæ required:

$$U = \xi\{\cos D_0 \sin \delta \cos(\alpha - A_0) - \sin D_0 \cos \delta \cos 2(\alpha - A_0)\}$$
$$+ \eta\{\tfrac{1}{2} \sin 2D_0 \sin \delta \sin(\alpha - A_0) - \tfrac{1}{2}(1 + \sin^2 D_0) \cos \delta \sin 2(\alpha - A_0)\}$$
$$+ B_1\{\sin D_0 \cos \delta - \cos D_0 \sin \delta \cos(\alpha - A_0)\}, \qquad (16)$$

$$V = \xi\{\cos D_0 \cos 2\delta \sin(\alpha - A_0) + \tfrac{1}{2} \sin D_0 \sin 2\delta \sin 2(\alpha - A_0)\}$$
$$+ \eta\{\tfrac{3}{4} \cos^2 D_0 \sin 2\delta - \tfrac{1}{4}(1 + \sin^2 D_0) \sin 2\delta \cos 2(\alpha - A_0) - \tfrac{1}{2} \sin 2D_0 \cos 2\delta \cos(\alpha - A_0)\}$$
$$+ B_1 \cos D_0 \sin(\alpha - A_0). \qquad (17)$$

The equations of condition for finding the components X, Y, Z of the solar motion, Δk, Δn and the galactic rotational constants A_1, B_1 and l_0 become

$$\mu_\alpha \cos \delta = X \sin \alpha - Y \cos \alpha + \Delta k \cos \delta + \Delta n \sin \alpha \sin \delta + U, \qquad (18)$$

$$\mu_\delta = X \cos \alpha \sin \delta + Y \sin \alpha \sin \delta - Z \cos \delta + \Delta n \cos \alpha + V, \qquad (19)$$

in which we suppose that the right-hand sides of (16) and (17) are substituted for U and V. In the equations used by Wilson and Raymond the terms involving ξ and η are omitted.

3. *The Systematic Corrections to the G.C. Proper Motions.*—We denote by $\Delta\mu_\alpha \cos \delta$ and $\Delta\mu_\delta$ the systematic corrections to be applied algebraically to the equatorial components of proper motion. Then

$$\Delta\mu_\alpha \cos \delta = -\Delta k \cos \delta - \Delta n \sin \alpha \sin \delta - U, \qquad (20)$$

$$\Delta\mu_\delta = -\Delta n \cos \alpha - V. \qquad (21)$$

As regards the numerical values of Δk and Δn derived by Wilson and Raymond we give in the subsequent calculations double weight to the galactic solutions for reasons touched upon in section 1. (The disagreement found between the two sets of values for Δk and Δn is undoubtedly due, mainly, to the omission of the ξ and η terms in the equatorial solutions.) Following Wilson and Raymond, we adopt 330° for the value of l_0 and $-0''{\cdot}25$ for the value of B_1. As regards A_1, the solution in galactic co-ordinates gives $0''{\cdot}26$, but the authors adopt Plaskett and Pearce's value of $0''{\cdot}33$; as I have shown in a recent paper *, there are objections to accepting Plaskett and Pearce's value, and for the purpose of the present paper I have preferred to use the straight mean of all the values (given by Wilson and Raymond in their Table VI) obtained from radial velocity investigations. This value is $0''{\cdot}30$, which is in good agreement with the value of $0''{\cdot}26$ actually found by the authors from the proper motions of the *G.C.* in their second solution. The values adopted in the subsequent calculations are then:

$$\Delta k = -0''{\cdot}25;\ \Delta n = +0''{\cdot}36;\ A_1 = +0''{\cdot}30;\ B_1 = -0''{\cdot}25;\ l_0 = 330°;\ A_0 = 190°;\ D_0 = 28°.$$

Table I gives, to two places of decimals, the centennial corrections $\Delta\mu_\alpha \cos \delta$ for the complete range of right ascension and for positive values of the declination.

For negative values of the declination the correction is obtained directly from Table I using $\alpha \pm 12^h$ for the right ascension and regarding the declination as positive.

For example, if $\alpha = 16^h$, $\delta = +50°$ and $\mu_\alpha \cos \delta$ (centennial) $= +13''{\cdot}4$, the table gives $\Delta\mu_\alpha \cos \delta = +0''{\cdot}26$, so that the corrected value of $\mu_\alpha \cos \delta$ is $+13''{\cdot}7$; if $\alpha = 1^h$, $\delta = -20°$ and $\mu_\alpha \cos \delta = -5''{\cdot}2$, the table gives, for $\alpha = 13^h$ and $\delta = +20°$, $\Delta\mu_\alpha \cos \delta$ to be $+0''{\cdot}30$, so that the corrected value of $\mu_\alpha \cos \delta$ is $-4''{\cdot}9$.

Table II gives, similarly, the corrections $\Delta\mu_\delta$ to be applied algebraically to the centennial values of μ_δ for the complete range in right ascension and for positive values of the declination.

* *M.N.*, 100, 370, 1940.

TABLE I

Values of the Correction $\Delta\mu_\alpha \cos\delta$

(*Unit*, 0″·01 *per century*)

α \ δ	0°	+10°	+20°	+30°	+40°	+50°	+60°	+70°	+80°	+90°
h										
0	+49	+54	+58	+61	+61	+60	+56	+51	+45	+37
1	+41	+45	+47	+48	+47	+46	+42	+37	+32	+25
2	+32	+34	+34	+34	+32	+30	+26	+22	+17	+12
3	+24	+23	+22	+20	+17	+14	+10	+ 6	+ 2	− 3
4	+20	+17	+13	+ 9	+ 5	0	− 4	− 9	−13	−17
5	+20	+15	+ 8	+ 3	− 4	−10	−16	−21	−26	−30
6	+25	+17	+ 9	+ 1	− 7	−15	−23	−30	−36	−40
7	+33	+24	+14	+ 4	− 6	−16	−26	−35	−42	−49
8	+42	+32	+21	+ 9	− 2	−14	−26	−36	−46	−54
9	+49	+39	+28	+15	+ 3	−10	−23	−34	−45	−55
10	+54	+44	+33	+20	+ 8	− 5	−18	−31	−42	−52
11	+53	+45	+34	+23	+11	− 1	−13	−25	−36	−46
12	+49	+42	+33	+24	+14	+ 3	− 7	−18	−28	−37
13	+41	+36	+30	+23	+15	+ 7	− 1	− 9	−17	−25
14	+32	+30	+26	+22	+17	+12	+ 6	0	− 6	−12
15	+24	+24	+24	+22	+20	+18	+14	+11	+ 7	+ 3
16	+20	+23	+24	+26	+26	+26	+24	+22	+20	+17
17	+20	+25	+29	+32	+34	+36	+36	+35	+33	+30
18	+25	+31	+37	+42	+45	+47	+48	+47	+44	+40
19	+33	+41	+47	+52	+56	+58	+58	+57	+54	+49
20	+42	+50	+57	+63	+66	+68	+67	+64	+60	+54
21	+49	+58	+65	+70	+73	+74	+72	+68	+62	+55
22	+54	+62	+68	+72	+74	+74	+72	+67	+60	+52
23	+53	+61	+66	+69	+70	+70	+66	+61	+54	+46
24	+49	+54	+58	+61	+61	+60	+56	+51	+45	+37

For negative declinations extract the entry in Table II for $\alpha \pm 12^h$ and the corresponding positive declination *and change the sign.*

For example, if $\alpha = 3^h$, $\delta = -50°$ and $\mu_\delta = -8''{\cdot}8$ the value from the table for $\alpha = 15^h$ and $\delta = +50°$ is $+0''{\cdot}50$, so that the correction is $-0''{\cdot}50$; the corrected value of μ_δ is then $-9''{\cdot}3$.

I am indebted to Mr. T. R. Tannahill, M.A., B.Sc., and Mr. Ian N. Sneddon, B.Sc., for their efficient assistance in preparing Tables I and II.

4. *The Equations of Condition in Galactic Co-ordinates.*—The equations for determining the solar motion, precessional and equinox corrections and the constants of galactic rotation from the proper motion components in galactic co-ordinates have been given by Van de Kamp and Vyssotsky.* These formulæ are given in terms of the precessional and equinox corrections Δp and $(\Delta e + \Delta\lambda)$, which are related to Δk and Δu by

$$\Delta p = \Delta n \operatorname{cosec} \epsilon, \tag{22}$$

$$\Delta k + \Delta e + \Delta\lambda = \Delta n \cot \epsilon, \tag{23}$$

where ϵ is the obliquity of the ecliptic, and in terms of the inclinations of the ecliptic and the equator to the plane of the galactic equator and the galactic longitudes of the ascending nodes of the ecliptic and equator on the galactic equator.

* *A.J.*, **45**, 167, 1936, and *Publ. Leander McCormick Obs.*, **7**, 7, 1937. See also Wilson and Raymond, *A.J.*, **47**, 57, 1938.

TABLE II

Values of the Correction $\Delta\mu_\delta$

(*Unit*, 0″·01 *per century*)

α \ δ	0°	+10°	+20°	+30°	+40°	+50°	+60°	+70°	+80°	+90°
h										
0	−24	−21	−20	−20	−22	−25	−29	−33	−37	−40
1	−25	−23	−23	−25	−28	−32	−37	−41	−46	−49
2	−24	−23	−23	−26	−30	−34	−40	−46	−50	−53
3	−22	−20	−20	−22	−26	−32	−38	−45	−50	−55
4	−18	−14	−14	−15	−19	−25	−32	−40	−47	−52
5	−13	− 8	− 5	− 6	− 9	−15	−22	−31	−39	−46
6	− 7	0	+ 3	+ 4	+ 2	− 3	−11	−19	−28	−37
7	− 1	+ 7	+12	+14	+12	+ 8	+ 2	− 7	−16	−25
8	+ 6	+13	+19	+22	+22	+19	+13	+ 6	− 3	−12
9	+12	+19	+24	+28	+29	+28	+24	+18	+10	+ 3
10	+17	+23	+28	+32	+34	+34	+32	+28	+23	+17
11	+21	+26	+31	+35	+37	+39	+39	+37	+34	+30
12	+24	+28	+32	+36	+40	+42	+44	+44	+43	+40
13	+25	+28	+32	+37	+41	+46	+49	+50	+50	+49
14	+24	+28	+32	+38	+43	+48	+52	+54	+55	+53
15	+22	+26	+32	+38	+44	+50	+54	+57	+57	+55
16	+18	+23	+30	+38	+45	+51	+55	+56	+56	+52
17	+13	+20	+28	+36	+44	+50	+53	+53	+51	+46
18	+ 7	+15	+24	+33	+40	+46	+48	+47	+43	+37
19	+ 1	+ 9	+19	+27	+34	+38	+39	+37	+32	+25
20	− 6	+ 3	+11	+19	+24	+27	+28	+25	+19	+12
21	−12	− 4	+ 3	+ 9	+13	+14	+13	+10	+ 4	− 3
22	−17	−11	− 6	− 2	0	0	− 2	− 6	−11	−17
23	−21	−17	−14	−12	−12	−14	−16	−20	−25	−30
24	−24	−21	−20	−20	−22	−25	−29	−33	−37	−40

The formulæ in terms of Δk, Δn and the equatorial co-ordinates (A_0, D_0) of the pole of the galactic equator can be derived readily in literal form by the method of section 2.

The components of solar motion in galactic co-ordinates and Oort's expressions for the effect of rotation on the values of $\mu_l \cos b$ and μ_b are written down in the usual way. To obtain the precessional effects we have to transform the equatorial components

$$U_1 \equiv \Delta k \cos\delta + \Delta n \sin\alpha \sin\delta,$$
$$V_1 \equiv \Delta n \cos\alpha$$

into galactic components, u_1 and v_1, by means of

$$u_1 = U_1 \cos\phi + V_1 \sin\phi,$$
$$v_1 = -U_1 \sin\phi + V_1 \cos\phi.$$

Hence

$$u_1 = \Delta k \cos\delta \cos\phi + \Delta n(\sin\alpha \sin\delta \cos\phi + \cos\alpha \sin\phi),$$
$$v_1 = -\Delta k \cos\delta \sin\phi + \Delta n(\cos\alpha \cos\phi - \sin\alpha \sin\delta \sin\phi).$$

In these equations we write

$$\sin\alpha \equiv \sin(\overline{\alpha - A_0} + A_0) = \cos A_0 \sin(\alpha - A_0) + \sin A_0 \cos(\alpha - A_0),$$

with a similar procedure for $\cos\alpha$.

Making use of the formulæ (8) to (15) we obtain, after simple reductions,

$$u_1 = \Delta k(\sin D_0 \cos b - \cos D_0 \sin l \sin b) + \Delta n\{\cos A_0 \cos l \sin b - \sin A_0(\cos D_0 \cos b + \sin D_0 \sin l \sin b)\}, \quad (24)$$

$$v_1 = -\Delta k \cos D_0 \cos l - \Delta n(\cos A_0 \sin l + \sin A_0 \sin D_0 \cos l). \quad (25)$$

These formulæ can be simplified for computational purposes as follows:—

Define a and θ by

$$a \sin \theta = \tan A_0 \sin D_0, \quad (26)$$

$$a \cos \theta = 1. \quad (27)$$

Also, let

$$\Delta K = \Delta k \cos D_0, \quad (28)$$

$$\Delta N = a\Delta n \cos A_0. \quad (29)$$

Then (24) and (25) become

$$u_1 = \Delta K(\tan D_0 \cos b - \sin l \sin b) + \Delta N\{\sin b \cos (l+\theta) - \cot D_0 \sin \theta \cos b)\} \quad (30)$$

$$v_1 = -\Delta K \cos l - \Delta N \sin (l+\theta). \quad (31)$$

From (26) and (27) we find, for $A_0 = 190°$ and $D_0 = 28°$,

$$\theta = 4°{\cdot}7, \quad (32)$$

$$a = 1{\cdot}0034. \quad (33)$$

Writing (30) and (31) in the form

$$u_1 = p\Delta K + q\Delta N,$$
$$v_1 = r\Delta K + s\Delta N,$$

the equations of condition for the determination of the solar motion, Δk and Δn, and the rotational constants become

$$\mu_l \cos b = X \sin l - Y \cos l + p\Delta K + q\Delta N + A_1 \cos 2(l - l_0) \cos b + B_1 \cos b, \quad (34)$$

$$\mu_b = X \cos l \sin b + Y \sin l \sin b - Z \cos b + r\Delta K + s\Delta N - A_1 \sin 2(l - l_0) \sin b \cos b. \quad (35)$$

Summary

1. The equations of condition for deriving the solar motion, the precessional corrections Δk and Δn, and the rotational constants A, B and l_0 from the components of proper motion are derived in literal form, involving the co-ordinates of the galactic pole, for equatorial co-ordinates.

2. Numerical tables are given for the corrections $\Delta\mu_\alpha \cos\delta$ and $\Delta\mu_\delta$ to the components of the proper motions of the *General Catalogue*.

3. Literal equations are given when the solution is effected in galactic co-ordinates.

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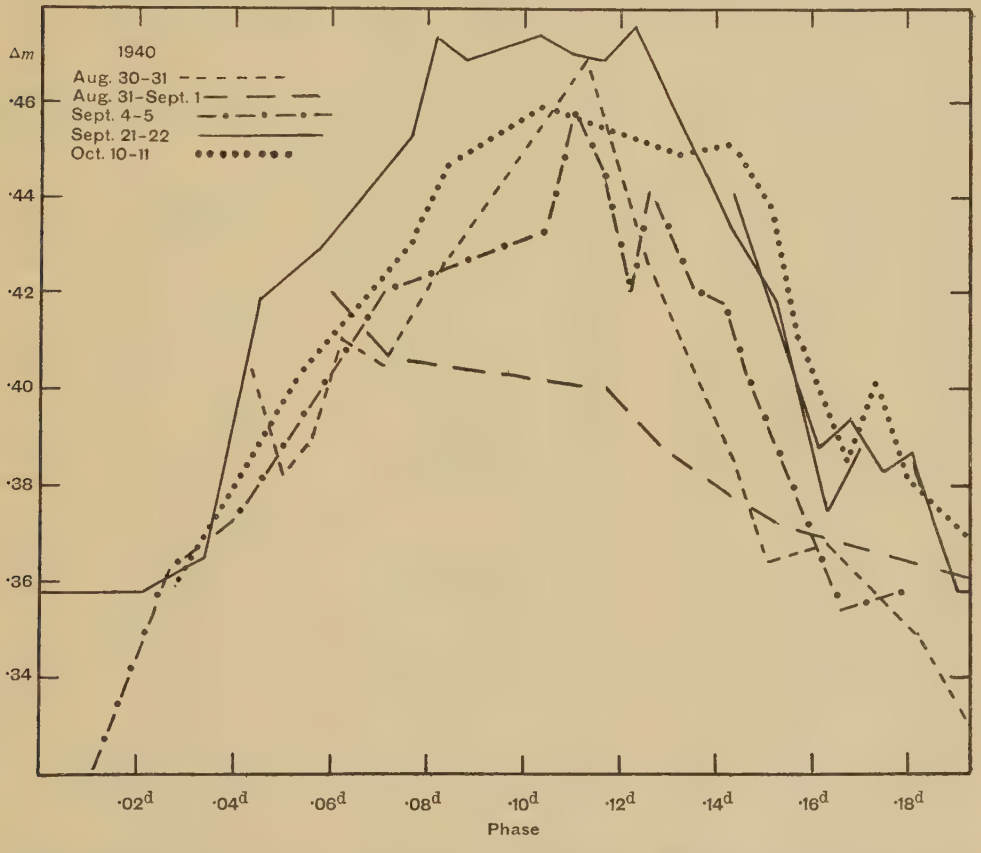
1940 *August* 30.

PHOTO-ELECTRIC OBSERVATIONS OF 12 LACERTAE

H. E. Green, M.A.

(Communicated by the Director, Cambridge Observatory)

The variable star 12 Lacertae (α 22^h 38^m, δ 39° 58′) has a period 4^h 38^m. It was discovered independently by Stebbins (1917) and Guthnick (1918), both of whom reported variations in the amplitude of the light-curve. The fluctuation of amplitude has been especially studied by E. A. Fath *, who made photo-electric observations in 1937 and 1939. In April 1940 Professor Fath wrote to Sir A. S. Eddington saying that he planned



a further series of observations at the Lick Observatory, beginning on August 15, and asking the Cambridge Observatory to co-operate, since the observations at Cambridge would be 8 hours earlier than at Lick.

Circumstances hindered the start, but the object-glass and mirror of the Sheepshanks telescope were re-mounted at the end of August. Observing conditions have been unusually favourable, and observations have been made on 20 nights from August 30 (J.D. 2429872) to October 29 (J.D. 2429932). Since the observations are intended to be combined with those of Professor Fath, no detailed discussion has been made. The variation of amplitude is, however, clearly seen in the diagram, which shows the

* *Popular Astronomy*, 46, No. 5, 1938.

light-curves obtained on 5 different nights. The full results are given in the table. The Julian date includes the heliocentric correction. The phase has been calculated from the formula communicated by Professor Fath in his letter:

$$\text{Max. J.D. } 2421439{\cdot}813 + 0{\cdot}19308815E.$$

The third column contains the magnitude determinations, the comparison star being 10 Lacertae. Corrections for atmospheric extinction have been applied, but these have not exceeded $0^{m}{\cdot}004$.

On some nights control observations were made on 9 Lacertae. The differences 9 Lac − 10 Lac (which should be constant) are given in the fourth column, since these may be of use in judging the genuineness of minor irregularities. It is to be noted, however, that these control observations are likely to be considerably less accurate than those of the variable itself, since owing to the greater distance between the stars the extinction corrections are about 10 times greater.

Julian Day	Phase	Δm 10 Lac − 12 Lac	Δm 9 Lac − 10 Lac	Julian Day	Phase	Δm 10 Lac − 12 Lac	Δm 9 Lac − 10 Lac	Julian Day	Phase	Δm 10 Lac − 12 Lac	Δm 9 Lac − 10 Lac
2429		−0·	+0·	2429		−0·	+0·	2429		−0·	+0·
872·4014	·043	404		879·4084	·099	443	174	886·5314	·077	461	
·4085	·050	382		·4175	·108	413	204	·5393	·085	451	
·4146	·056	390		·4265	·117	427	191	·5461	·092	473	
·4213	·063	410		·4361	·126	436	201				
·4301	·071	405		·4628	·153	366		890·3713	·055	385	
·4407	·082	424		·4710	·161	339		·3882	·072	411	
·4715	·113	469	184					·4076	·092	438	
·4843	·126	426	219	880·4812	·013	397		·4161	·100	451	
·5022	·144	385	223	·4899	·021	402					
·5101	·151	364	217	·4961	·028	379		891·3955	·114	469	
·5218	·163	368	201	·5019	·034	372		·4027	·121	454	
·5410	·182	349	205	·5092	·041	383		·4113	·130	438	
·5553	·004	325		·5153	·047	384					
								894·3215	·144	441	
873·3840	·060	420	204	882·4000	·001	347		·3389	·161	388	
·3958	·072	407	224	·4072	·008	362		·3458	·168	394	
·4415	·117	400	222	·4336	·034	390		·3522	·175	383	
·4518	·128	389	218	·4444	·045	396		·3589	·181	387	
·4664	·142	379	210	·4610	·062	444		·3685	·191	358	
·5109	·187	362	204	·4667	·067	478		·3922	·022	358	
·5267	·009	361	189	·4916	·092	467		·4053	·035	365	221
				·5183	·119	461		·4157	·045	419	223
877·3896	·011	321	218	·5240	·125	425		·4273	·057	429	204
·4054	·026	361	225	·5309	·132	440		·4474	·077	453	
·4185	·040	373		·5371	·138	423		·4530	·082	474	
·4499	·071	420		·5428	·143	405		·4589	·088	469	
·4830	·104	433		·5492	·150	422		·4737	·103	474	
·4886	·110	458		·5546	·155	399		·4815	·111	470	
·4952	·116	446						·4869	·116	469	
·5009	·122	420		886·3995	·138	423		·4934	·123	476	
·5071	·126	441		·4080	·147	396		·5085	·138	445	
·5135	·135	421		·4334	·172	379		·5137	·143	434	
·5197	·141	418		·4451	·184	387		·5232	·152	418	
·5257	·147	402		·4591	·005	363		·5337	·163	375	
·5450	·166	354	226	·4654	·011	379		·5404	·170	388	
·5579	·179	358	220	·4713	·017	403					
·5712	·192	369	205	·4950	·041	388	214	896·4031	·102	469	
·5839	·012	342	214	·5062	·052	438	225	·4086	·107	454	
·5949	·023	350	205	·5191	·065	447	229	·4146	·113	454	

Julian Day	Phase	Δm 10 Lac − 12 Lac	Δm 9 Lac − 10 Lac
2429		−0·	+0·
896·4218	·120	463	
·4463	·145	437	
·4557	·154	402	
·4772	·166	375	
·4846	·183	386	
898·3055	·073	447	
·3189	·086	442	
·3259	·093	473	
·4082	·176	387	
·4144	·182	390	
·4201	·188	408	
·4251	·193	393	
·4512	·026	413	220
·4655	·040	380	
·4726	·047	393	
903·3811	·128	468	
904·3849	·167	374	
·3902	·172	364	
·3958	·178	371	
2429		−0·	+0·
910·3428	·140	420	
·3510	·147	436	
913·3214	·027	359	
·3471	·053	402	
·3557	·062	413	
·3709	·077	431	
·3775	·084	446	
·3983	·104	459	
·4264	·132	449	
·4378	·144	451	
·4449	·151	438	
·4522	·157	410	
·4610	·167	385	
·4670	·173	401	
·4739	·180	381	
·4901	·004	368	
·4986	·012	358	
917·4477	·100	469	
·4529	·105	459	
·4597	·112	466	
2429		−0·	+0·
917·4660	·118	432	
·4724	·124	432	
·4856	·137	474	
·4913	·143	432	
923·2903	·149	389	
927·3018	·106	431	
·3167	·121	450	
·3226	·127	424	
·3292	·134	421	
·3355	·140	421	
·3432	·147	417	
·3486	·153	411	
·3750	·179	373	234
·3827	·187	382	236
·3919	·003	371	232
·3999	·011	379	232
932·3442	·128	442	
·3529	·137	423	
·3613	·145	428	

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH

(Communicated by the Astronomer Royal)

The following list is a continuation of the relative parallaxes obtained with the 26-inch refractor, and published annually in the November number of *Monthly Notices*.

The results are given in the same form as in previous lists.

The mean probable error for unit weight is ±″·023, and the average number of plates per star is 28.

No.	Name and B.D. No.	Mag. Vis. Phot.	Spec. Type	R.A. Dec. 1900·0	Unit 0″·001				
					Proper Motion		P.E. Unit Weight	Parallax	P.E.
					R.A.	Dec.			
				h m s					
1	P. I 88	7·0	Ko	1 25 09	+109	− 50	20	− 4	± 6
	67° 133	7·8		+67° 54′	+ 98				± 2
2	P. II 86	6·7	Ko	2 24 53	+ 64	− 81	17	+ 4	± 4
	70° 183	7·5		+70° 51′	+ 46				± 2
3	Br. 344	5·9	Ko	2 33 21	+ 14	− 93	23	+ 9	± 8
	80° 86	7·0		+81° 01′	+ 5				± 4
4	Gr. 594	7·0	Ko	3 01 27	− 16	− 83	21	− 3	± 6
	79° 94	7·7		+79° 45′	− 25				± 7
5	O.A. 3646	9·3	Ko	3 13 21	+ 92	− 53	21	+ 10	± 6
	64° 386	9·1		+64° 40′	+ 79				± 4
6	A.C. 72° 1831	...		3 25 26	+170	−111	25	+ 13	± 7
		11·1		+72° 52′	+166				± 8
7	C. 539	7·2	Ko	3 45 29	+ 77	− 83	19	+ 2	± 5
	81° 134	8·0		+81° 35′	+ 48				± 3
8	A.C. 67° 1401	...		4 30 00	+242	−276	19	+ 10	± 5
		11·2		+67° 24′	+216				± 4
9	73° 257′	...		4 44 33	+ 89	−183	19	+ 6	± 5
		9·6		+73° 41′	+ 90				± 4
10	L. 10769	6·4	Ko	5 42 10	+ 26	− 45	25	0	± 7
	68° 412	6·8		+68° 27′	+ 6				± 4
11	A.C. 82° 1111	...		5 49 07	+114	−1291	20	+105	± 5
		11·7		+82° 08′	+ 90				± 3
12	A.C. 67° 2081	...		6 10 32	− 48	−208	25	− 2	± 6
		12·1		+67° 46′	− 61				± 6
13	73° 609	8·7	Ko	13 55 36	− 97	−136	22	+ 17	± 7
		9·1		+73° 21′	− 91				± 6
14	O.A. 14987	7·9	Ko	14 52 33	+ 53	− 34	22	+ 21	± 6
	65° 1024	8·5		+65° 03′	+ 47				± 4

No.	Name and B.D. No.	Mag. Vis. Phot.	Spec. Type	R.A. Dec. 1900·0	Unit 0″·001 Proper Motion R.A.	Proper Motion Dec.	P.E. Unit Weight	Parallax	P.E.
				h m s					
15	A.C. 69° 6829	...		17 13 52	− 59	−342	30	− 7	± 9
		13·2		+69° 39′	− 78				± 5
16	A.C. 64° 5510	...		17 48 55	+220	−137	28	− 2	± 9
		11·1		+64° 25′	+223				± 5
17	A.C. 77° 6537	...		18 32 48	+ 38	+255	29	+ 5	±11
		12·3		+77° 59′	+ 32				± 8
18	A.C. 75° 7125	...		18 47 48	− 97	+177	22	+ 3	± 7
		11·3		+75° 53′	−112				± 5
19	L.F. 3291	...	Ko	19 47 22	− 36	+ 53	20	+ 8	± 6
	65° 1404	9·5		+65° 12′	− 38				± 3
20	74° 859	8·9	Ko	20 24 21	+ 5	+ 61	23	− 12	± 7
		9·5		+75° 07′	− 6				± 2
21	P. XX 359	6·5	Ko	20 43 57	− 25	− 38	22	− 2	± 7
	69° 1127	7·1		+69° 23′	− 26				± 2
22	A.C. 80° 5262	...		21 10 28	+266	+157	22	+ 38	± 8
		11·8		+80° 23′	+265				± 4
23	A.C. 74° 8059	...		22 32 55	+244	+ 73	30	− 3	±10
		12·7		+74° 10′	+266				± 6
24	Br. 3058	5·0	K5	22 55 13	+ 93	+ 25	25	− 3	± 8
	83° 640	6·0		+83° 49′	+ 86				± 3
25	L.F. 4384	7·6	Ko	23 03 43	+ 43	− 25	26	+ 10	± 9
	69° 1307	8·1		+70° 07′	+ 50				± 3
26	L. 45571	8·0	Ko	23 10 17	+ 54	+ 57	24	− 1	± 7
	73° 1021	8·5		+74° 00′	+ 50				± 2
27	Gr. 4070	7·0	Ko	23 22 54	− 7	− 69	19	− 12	± 8
	64° 1810	7·7		+65° 04′	− 12				± 5
28	Br. 3166	5·0	Ao	23 43 08	+ 22	− 7	31	+ 24	± 9
	67° 1562	5·2		+67° 15′	+ 29				± 4

MICROMETRICAL MEASURES OF DOUBLE STARS

Rev. T. E. R. Phillips, M.A.

The following measures have been made with the 8-inch refractor lent by the Society:—

Names	R.A. 1950·0	Dec. 1950·0	Mags.	Position Angle	Distance	No. of Nights	Date 1900 +
	h m	° ′		°	″		
Σ 60 (η Cas)	0 46·0	+57 33	3·7, 7·4	278·5	9·13	4	38·72
,,	...	...	...	279·1	9·35	3	39·91
Σ 73 (36 And)	0 52·2	+23 21	6·2, 6·8	112·7	0·57	8	38·75
Σ 186	1 53·3	+ 1 35	7·2, 7·2	45·5	1·40	3	38·78
Σ 285	2 35·9	+33 12	7·0, 7·7	166·2	1·69	3	38·80
Σ 425	3 37·0	+33 58	7·3, 7·3	82·9	2·17	3	38·81
Σ 572	4 35·4	+26 51	6·5, 6·5	196·4	3·89	5	38·83
OΣ 95	5 02·6	+19 44	7·0, 7·6	315·5	0·89	3	38·81
OΣ 98 (14 Ori)	5 05·2	+ 8 26	5·9, 6·7	108·8	0·87	3	38·81
Σ 644	5 06·9	+37 13	6·7, 7·0	221·9	1·72	3	38·81
Σ 774 (ζ Ori)	5 38·3	− 1 57	2·0, 5·7	160·2	2·51	4	40·16
Σ 795 (52 Ori)	5 45·3	+ 6 27	6·2, 6·2	210·1	1·44	4	37·88
Σ 948 AB (12 Lyn)	6 41·8	+59 32	5·2, 6·1	98·0	1·74	4	38·98
,, AC	...	...	5·2, 7·4	307·9	8·54	4	38·98
,, BC	...	...	6·1, 7·4	302·3	10·44	4	38·98
Σ 1037	7 09·7	+27 20	7·1, 7·1	342·4	0·76	4	38·24
,,	...	...	...	340·7	0·89	3	39·17
OΣ 170	7 14·9	+ 9 24	7·5, 7·5	98·7	1·55	3	38·99
Σ 1110 (*Castor*)	7 31·4 ...	+32 00	2·7, 3·7	203·4	3·80	5	38·03
,,	...	...	...	202·5	3·54	3	38·90
OΣ 175	7 32·0	+31 04	6·0, 6·6	328·9	0·55	5	38·11
Σ 1196 AB (ζ Cnc)	8 09·2	+17 48	5·0, 5·7	100·7	0·72	8	38·14
,, ,,	...	...	...	95·4	0·69	8	39·04
,, AC	...	...	5·0, 5·5	101·4	6·14	8	38·22
,, ,,	...	...	...	100·6	6·18	4	39·06
,, BC	...	...	5·7, 5·5	101·7	5·30	5	38·24
,, ,,	...	...	...	102·1	5·36	4	39·06
Σ 1291	8 51·2	+29 46	5·9, 6·4	318·4	1·51	3	38·92
Σ 1306 (σ^2 UMa)	9 06·0	+67 21	5·0, 8·2	58·8	1·62	7	38·26
Σ 3121	9 14·9	+28 47	7·5, 7·8	27·0	0·66	4	38·20
,,	...	...	...	28·8	0·58	7	39·17
Σ 1355	9 25·1	+ 6 28	7·2, 7·2	341·2	2·56	4	38·23
Σ 1356 (ω Leo)	9 25·7	+ 9 17	6·0, 7·0	142·5	0·87	6	38·14
,,	...	...	...	143·5	0·80	9	39·15
OΣ 215	10 13·6	+17 59	7·0, 7·2	193·0	1·07	3	38·17
,,	...	...	...	194·2	1·10	3	39·28
Σ 1424 (γ Leo)	10 17·2	+20 06	2·6, 3·6	118·9	3·99	6	39·12
Σ 1523 (ξ UMa)	11 15·5	+31 50	4·0, 4·9	286·7	1·79	13	38·26
,,	...	...	...	278·8	1·78	9	39·23
OΣ 235	11 29·6	+61 23	6·0, 7·3	24·2	0·80	4	38·24
Σ 1639	12 21·9	+25 51	6·7, 7·9	333·8	0·99	7	38·23
,,	...	...	...	333·1	1·02	3	39·17

Names	R.A. 1950·0	Dec. 1950·0	Mags.	Position Angle	Distance	No. of Nights	Date 1900 +
	h m	° ′		°	″		
Σ 1647	12 27·9	+ 9 59	7·5, 7·8	229·4	1·19	3	39·32
Σ 1670 (γ Vir)	12 39·2	− 1 11	3·6, 3·7	316·8	5·72	4	38·18
,,	...	...	...	317·6	5·66	3	39·17
Σ 1687 (35 Com)	12 50·8	+21 31	5·0, 7·8	120·2	1·01	4	38·38
,,	...	...	...	118·4	0·97	3	39·20
Σ 1816	14 11·7	+29 20	7·0, 7·1	83·0	1·28	4	38·29
,,	...	...	...	84·2	0·89	4	39·43
Σ 1834	14 18·5	+48 44	7·1, 7·2	99·1	0·69	3	38·27
,,	...	...	...	98·2	0·74	3	39·25
Σ 1865 (ζ Boo)	14 38·8	+13 57	4·4, 4·8	132·9	1·10	4	38·23
,,	...	...	...	132·8	1·16	4	39·23
Σ 1877 (ϵ Boo)	14 42·8	+27 17	3·0, 6·3	334·2	2·75	3	38·32
Σ 1888 (ξ Boo)	14 49·0	+19 18	4·7, 6·6	12·0	4·94	5	38·23
,,	...	...	...	9·9	5·06	3	39·23
OΣ 288	14 51·1	+15 54	6·4, 7·1	180·7	1·70	3	38·29
,,	...	...	...	181·3	1·65	4	39·38
Σ 1909 (44 Boo)	15 02·2	+47 50	5·2, 6·1	248·1	2·61	3	38·61
Σ 1937 (η CrB)	15 21·1	+30 27	5·2, 5·7	314·7	0·42*	6	38·27
,,	...	...	...	334·7	0·45*	3	39·36
Σ 1938	15 22·6	+37 31	6·7, 7·3	35·5	1·78	3	38·30
OΣ 303	15 58·6	+13 27	7·4, 7·9	156·5	0·98	3	38·34
Σ 1998 AB (ξ Sco)	16 01·6	−11 15	4·9, 5·2	27·5	1·06	5	38·40
,, ,,	...	...	...	28·0	1·08	4	39·32
,, AC	...	...	4·9, 7·2	54·3	7·94	3	38·40
,, ,,	...	...	...	53·9	8·11	5	39·47
,, BC	...	...	5·2, 7·2	60·2	6·91	4	38·40
,, ,,	...	...	...	61·0	6·92	3	39·47
Σ 2052	16 26·6	+18 29	7·5, 7·5	230·5	0·66	4	38·32
,,	...	...	...	227·5	0·57	6	39·32
Σ 2084 (ζ Her)	16 39·5	+31 39	3·0, 6·5	203·7	1·15	4	38·29
,,	...	...	...	192·7	0·99	3	39·39
Σ 2130 (μ Dra)	17 04·3	+54 32	5·0, 5·1	101·8	2·27	7	38·50
,,	...	...	...	101·2	2·22	3	39·67
Σ 2140 (α Her)	17 12·3	+14 26	3·0, 6·1	109·9	4·76	3	39·42
Σ 2173	17 27·9	− 1 01	5·8, 6·1	337·9	0·94	3	38·65
OΣ 338	17 49·7	+15 19	7·2, 7·4	359·2	0·68	6	39·34
Σ 2272 (70 Oph)	18 02·9	+ 2 33	4·1, 6·1	116·6	6·82	4	38·45
,,	...	...	...	116·0	6·62	4	39·65
Σ 2289	18 07·7	+16 27	6·0, 7·1	227·2	1·19	5	39·53
OΣ 358	18 33·6	+16 56	6·7, 7·1	178·5	2·03	5	38·47
Σ 2382 (ϵ^1 Lyr)	18 43·8	+39 37	5·1, 6·1	5·4	2·91	3	38·40
Σ 2383 (ϵ^2 Lyr)	...	...	5·1, 5·4	110·6	2·34	3	38·40
Σ 2525	19 24·5	+27 12	7·4, 7·6	297·9	1·17	4	39·58
Σ 2579 (δ Cyg)	19 43·4	+44 59	3·0, 7·9	258·2	1·96	4	38·97
,,	...	...	...	261·5	1·92	5	39·56
OΣ 395 (16 Vul)	19 59·9	+24 47	5·8, 6·2	110·9	0·84	4	38·76
OΣ 413 (λ Cyg)	20 45·4	+36 13	5·0, 6·3	40·1	0·74	3	38·49
Σ 2950	22 49·4	+61 25	5·7, 7·0	300·2	1·91	3	38·67
Σ 3050 (37 And)	23 57·0	+33 28	6·0, 6·0	244·8	1·76	3	38·64

* Notched only—not really divided.

Erratum

Dr. A. G. Walker's paper, *M.N.*, **100**, No. 8:

P. 627, section 6, *for* $k = -1$ *read* $k = 1$. Section 7, *for* $k = 1$ *read* $k = -1$.